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Physics. — *Second order disturbance terms (*BROWNE *terms) in pendulum observations at sea.* By F. A. VENING MEINESZ.

(Communicated at the meeting of May 28, 1938.)

Since the first paper on this subject communicated in the September meeting of 1937, the writer made the trip announced in that paper, with Hr. Ms. Submarine O 12 from Curaçao to Holland. The results obtained and the trouble experienced with the provisional instrument for measuring the horizontal accelerations of the ship made it possible to draw up the plans for a more durable shape of the apparatus and this has since then been constructed. The Navy gave the opportunity to test it during a trip of one week in the beginning of this month and as it has proved to be successful, it will be adopted as final.

During both trips results have been obtained for the ship's movements during submergence and these data have allowed the computation of the corrections for the BROWNE terms for the old observations of gravity at sea. This paper gives some particulars about these trips, about the apparatus and about the results for the ship's movements, and it further contains the list of corrections to the results of 454 gravity stations at sea, which have been published by the Netherlands Geodetic Commission in: "Gravity Expeditions at Sea, 1923—1932", Waltman, Delft. Sincere thanks may be expressed to the authorities of the Netherlands Navy for the cooperation which she has again been willing to give for these researches.

In the end of October 1937 the writer left Holland for the island of Curaçao where he arrived on November 16. The time before the sailing of the submarine on November 26 was used for experimenting with the long period pendulum constructed for the determination of the horizontal accelerations of the ship. He wishes to express his thanks to Mr. VAN NYMEGEN SCHONEGEVEL, Managing Director of the great Oil Refining Plant of the C.P.I.M. on the island, for the facilities given to him for these experiments in the work-shop of the company.

The programme of a trial trip of a few days and of the main submarine voyage, and the nmber of gravity stations are given in the table on the following page.

In nearly all the stations the gravity observations were followed by a long programme of experiments with the long period pendulum for different ship's courses and often this was repeated at a greater depth of submergence; this generally took together about 3 or 4 hours submer-

		In	Curaçao	1 station
22	November	Departure	Curaçao	
23	„	Arrival	Curaçao	3 „
26	„	Departure	Curaçao	
29	„	Arrival	San Juan (Porto Rico)	4 „
1	December	Departure	San Juan	
15	„	Arrival	Punta Delgada (Azores)	12 „
17	„	Departure	Punta Delgada	
24	„	Arrival	Den Helder	2 „
			In total	22 stations
			(of which 21 new stations)	

ging. Because of this the number of gravity observations could not be as great as during former expeditions. The writer feels kindly indebted to the Captain, Lieut.-Commander H. C. W. MOORMAN for the opportunities he gave for the scientific work and also for his helpful assistance; he allotted the continuous assistance of an able sergeant torpedoist, PLAS, for helping him along with the instrumental work. The writer wishes likewise in this connection to mention with gratitude the names of the other members of the staff, Lieuts. C. A. JEEKEL and J. B. KENNEMA and the Chief of the Engine-room, Lieut. A. DE VAAL. With the exception of two days of heavy gale shortly before Punta Delgada the whole voyage has been favoured by fairly good weather.

The long period pendulum, which was installed on the pendulum apparatus for measuring the horizontal accelerations of the ship, has given many difficulties. It consisted of a horizontal beam with two weights at the ends, suspended by two tiny springs; the centre of gravity was adjustable by means of three screws, one for the movement in vertical sense and two for that in horizontal sense. In principle it worked all right; while the pendulum apparatus in its gimbals followed the apparent fluctuation of the direction of gravity as it is caused by the fluctuating horizontal accelerations of the ship which are added vectorially to the acceleration of gravity, the long period pendulum only underwent this fluctuation slightly in opposite sense and so the angle between the last pendulum and the pendulum apparatus in the gimbals, which was recorded photographically, gives data for deriving the horizontal accelerations. Difficulties arose, however, from the fact that, in making the springs strong enough for carrying the weight of the pendulum, they play an important part in the movement of the pendulum and this has the consequence, when the pendulum is adjusted in such a way that the period is great enough for the purpose, to make the period dependent on the amplitude and also variable because of the frequently occurring of deformations of the springs beyond the elastic limit. For many observations the result was an imperfect knowledge of the period of the pendulum and as this period enters in the formula's for the determination of the horizontal accelerations, uncertainties were brought about.

For several observations, however, the results for the horizontal accelerations were reliable enough for making sure that, at least approximately, the horizontal acceleration of the ship was equal to the vertical acceleration, independent of the direction of the wave-movement with regard to the course of the ship. This seems to prove two things. In the first place that in those cases the movement of the water-particles was at least approximately circular, which would confirm the theory of GERSTNER about the wave movement in the oceans. In the second place that the ship is carried along entirely by the water movement, even incase the course is parallel to the wave movement. This last point can obviously only be true incase the wave-length is not too small with regard to the length of the ship, which is 60 meters from bow to stern, but as the amplitude of short waves is decreasing much quicker with the depth of submerging than that of long waves — according to the rule of Rankine the amplitudes are halved for a depth of $^1/_9$ of the wave-length — this is practically true for the depths that were used of 20 meters or more; the periods of the vertical and horizontal accelerations at these depths are confirming this. The accurate study of the results and also that of the more satisfactory results of the last trip will show whether this conclusion is accurately true or whether there occur cases in which the carrying along of the ship shows differences for different orientations with regard to the wave movement.

The fact that the ship is entirely carried along, also when the axis is parallel to the wave movement, was a surprise to the writer; in the first communication on this subject he emitted the opinion that this would probably not be the case. This opinion was founded on his experience that submarines during submergence do not show any perceptible pitching of the axis, even when the sea at that depth shows strong movements. The explanation of this apparent contradiction is simple: the pitching is steered away by the men who operate the vertical rudders. During the last trips the writer gave special attention to the question and he found that at the same time when the long period pendulum showed evidences of strong waves coming, the rudders were put into action.

The difficulties experienced with the spring suspension during the voyage of the O 12 gave rise after returning home and after discussing the question also with Mr. BROWNE to replace the springs by knife-edges. A plan was drawn up for a final shape for the apparatus in such a way that it would form a harmonious whole with the original pendulum apparatus; it was planned to fit in the space between the pendulum apparatus and the recording apparatus. It is provided with two long period pendulums, one swinging parallel and one perpendicular to the main pendulums; this allows to get the two horizontal components of the ship's accelerations at the same time. These pendulums are recorded by means of the same source of light as the other pendulums and on the same record; the image of the light on the record is of smaller dimensions

than for the other recording rays and so the curves are thinner and more concentrated, the object of this was to avoid the obscuring of the other records. The pendulums are again provided with screws for adjusting the centre of gravity in vertical and horizontal sense. They are lifted from their knife-edges after the observation is finished and they are clamped for preventing their swinging uncontrolled during the intervals between the observations. They can be clutched during the observations for reducing their amplitudes if necessary; this can be done by means of lightly balanced levers operated by their own weight. As the distance between the centre of gravity of the pendulums and their axis of rotation is only about 25 μ, it may be foreseen that slight changes of temperature might cause perceptible changes of period and so it was considered desirable to be able to check the period during the observations. This can be done by means of dropping a light spherical weight in a receptacle attached to the pendulum at a horizontal distance of about one cm from the centre of gravity; the resulting displacement of the axis of the pendulum-record allows the computation of het period of the pendulum. The pendulums are swinging in normal atmospheric pressure; the damping is so slight that the coupling caused by this damping between the pendulum and the box in which it swings is probably negligible; if not, the effect can easily be taken into account in the formula's for the determination of the horizontal accelerations.

The new apparatus has been constructed in the work-shop of the Meteorological Institute at De Bilt, for which the Chief Director, Prof. VAN EVERDINGEN kindly gave his consent. It was done by the instrument-maker D. VAN LUNTEREN, whose diligence the writer wishes here to acknowledge; he took much trouble to get the instrument ready in the short time that was available. The Navy gave the opportunity to put it to the test by allotting Hr. Ms. Submarine O 13 for the time that was asked for the experiments. As it was desirable not to limit the observations to the North Sea where the wave-movement must be different from the movement in the oceans, the trip was extended to a point well out in the Atlantic at 47° 9′ Latitude and 10° 2′ Longitude west. The trip thus took one week, i.e. from May 3 till May 10. Observations were made at seven stations, two in the North Sea, one in the Channel and four near to the edge of the continental shelf, i.e. two over deep water and two over a depth of 130—150 meters. The writer was accompanied by Dr. W. NIEUWENKAMP, attached to the Netherlands Geodetic Commission, who in the future will undertake the maritime gravity research.

The trip was successful, the sea was rough enough for showing water movements during submergence, but still the trip was not too uncomfortable. The writer wishes to acknowledge the helpful assistance of the Captain, Lieut.-Commander G. B. M. VAN ERKEL, and also of the other members of the staff, Lieuts. J. W. CASPERS, W. J. SNYDER, A. OHR, in command of the engine-room, and W. D. J. GESTEL. At all the stations,

save those in the North Sea, measurements were made in all courses at two depths, viz. 20 meters and 40 meters. At the first station in the North Sea the same was done at one depth, 20 m, at the second it was done at two depths 12 m and 20 m. At all the stations gravity determinations were made. For the full programme it was necessary to submerge during three to four hours.

The new instrument has given full satisfaction. The long period pendulums were lowered on their knife-edges at the beginning of the observations and no further attendance was needed; they adopted only from time to time slight amplitudes of their own, which disappeared soon, and so their amplitudes were usually small or negligible. It was, therefore never necessary to clutch them for reducing their amplitudes. In this way the addition of this instrument to the old apparatus does not render the observations more difficult to handle.

The records have yet to be studied but a provisional investigation confirms the results of the previous trip: for every wave separately the ship's movement appears to be, at least approximately, circular in a vertical plane. This is shown by the resultant horizontal acceleration being about equal to the vertical acceleration but differing $\frac{1}{2}\pi$ in phase. Even in the North Sea, for a sea-depth of 40 meters, this seems to be near the truth. Of course a careful study will have to be made before any final and precise statement can be made.

The records obtained at 20 meters depth are generally much more complicated than those got at 40 meters depth; an instance of this is given by the parts that are reproduced. The computation of the second order correction for the gravity determination is, therefore, simpler for greater depth of submerging, where the correction is also smaller, and so when the sea is not quiet, a greater depth is always an advantage for the gravity observations. The greater complication of the record at 20 m is caused by the presence of shorter waves which have practically disappeared at a depth of 40 m. When we derive the wave-length λ from the period T according to GERSTNER's wave theory by means of the formula

$$T = \sqrt{\frac{2\pi\lambda}{g}}$$

the formula of Rankine about the decrease of the amplitude with the depth seems to be confirmed. It is, however, a curious fact that the wave-length which is thus obtained is invariably longer than what was estimated at the surface of the sea; it may even be $1\frac{1}{2}$ times more. This appears to be a fact which has often been found in oceanographic wave-studies and Ir. J. TH. THIJSSE, Director of the Hydraulic Laboratory at Delft, who gave this information, told the writer that it is usually attributed to errors in the estimating of the wave-length at the sea surface. Also about

these points further investigation will be necessary before definite data can be given.

For a thorough study of the ship and wave-movements the writer wishes to recommend the new apparatus that Mr. BROWNE is constructing. This is especially planned for this purpose while the object of the writer was merely to construct an instrument to complete the pendulum apparatus, which can easily be combined with it and of which the principal aim is to provide the data for computing the second order corrections of the gravity determinations at sea; future copies of the pendulum apparatus will be provided with it.

The formula for deriving the horizontal acceleration of the ship in the direction of the swinging plane of the long period pendulum from the record of this pendulum can be easily deduced by considering the effect on the pendulum and on the apparatus in the gimbals of a periodical horizontal acceleration $\ddot{y} = a \cos n_w t$ as it is given by a wave-movement of a half-period $T_w (n_w \; T_w = \pi)$.

The equation of motion of the apparatus in the gimbals is

$$\ddot{\theta}_a + n_a^2 \theta_a + \frac{n_a^2}{g} \ddot{y} = 0$$

in which $n_a = \pi / T_a$, T_a being the half-period of this system.

The solution is

$$\theta_a = p_a \cos n_w t + \text{ terms of the period } 2T_a$$

in which

$$p_a = -\frac{1}{1 - \frac{T_a^2}{T_\omega^2}} \cdot \frac{a}{g}.$$

The ratio T_a^2 / T_w^2 is considerably smaller than 1; the period $2\,T_a$ is 3 or 4 seconds and the period $2\,T_w$ mostly larger than 7 seconds, often (in the oceans) 10 to 13 seconds. As the ratio $-a/g$ is the amplitude of the apparent fluctuation of gravity, we see that the apparatus makes a slightly larger fluctuation with regard to a fixed system of coordinates. Superposed on this it can of course swing with its own period $2\,T_a$.

The equation of motion of the long period pendulum is

$$\ddot{\theta}_s + n_s^2 \theta_s + \frac{n_s^2}{g} \ddot{y} = 0$$

in which $n_s = \pi / T_s$, T_s being the half-period of this pendulum.

The solution is

$$\theta_s = p_s \cos n_w t + \text{ terms of the period } 2\,T_s$$

Parts of a record at Station N⁰. 4 (scale 1/1); Latit. 47° 9′ N, Longit. 10° 2′ W, sea-dept
records (the upper: right and middle pendulums combined, the lower: middle and left pendulum
record (broadened by slight vibrations of the ship); 2⁰. long period pendulum, parallel to ship's
plane of the main pendulums. The records rea

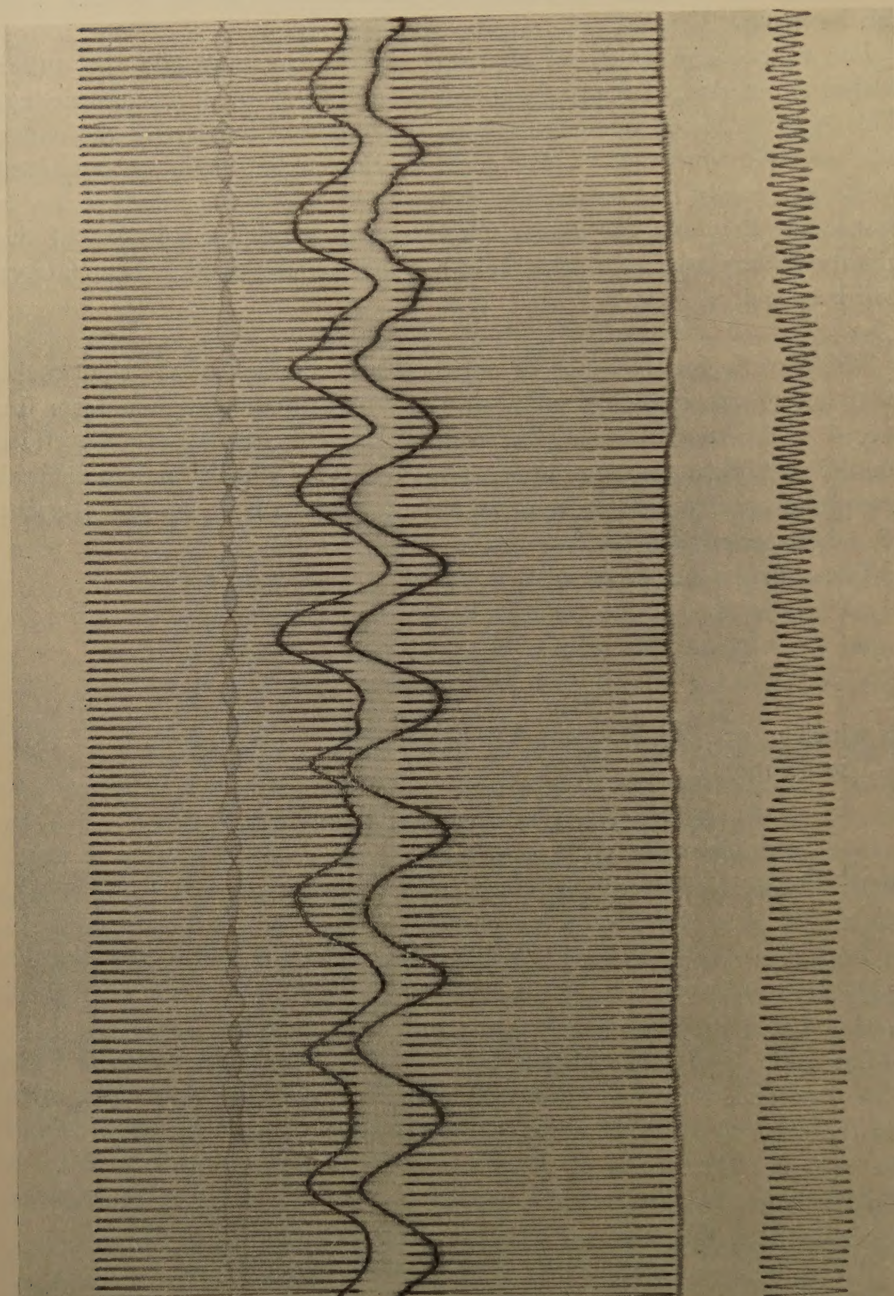

Fig. 1.

Record at a depth of 37 m (keel at 40 m); course slowly changing from N 60° E (right) to N 50° E (left); wave-period 14 sec.; wave-length according to Rankine 310 m; waves coming according to the records of the long period pendulums from N 280° E.

The fluctuations of the chronometer-marks correspond to the ship's velocity in vertical sense; they are in phase, resp. contrary phase with the records of the long period pendulums; the vertical accelerations, therefore, show a phase-difference of $\frac{1}{2}\pi$ from the horizontal accelerations.

Double amplitude of the wave, as deduced from the accelerations, about 80 cm; according to Rankine double amplitude of the wave at the surface about 180 cm; this long wave has not been noticed at the surface.

. The upper and middle regular curves, with white chronometer-marks, are the main pendulum
ower curve: the middle pendulum alone. Other records: from above to below: 1⁰. temperature
period pendulum, perpendicular to ship's axis; 4⁰. record of the position of the swinging-
left; the reproductions have to be turned 90°.

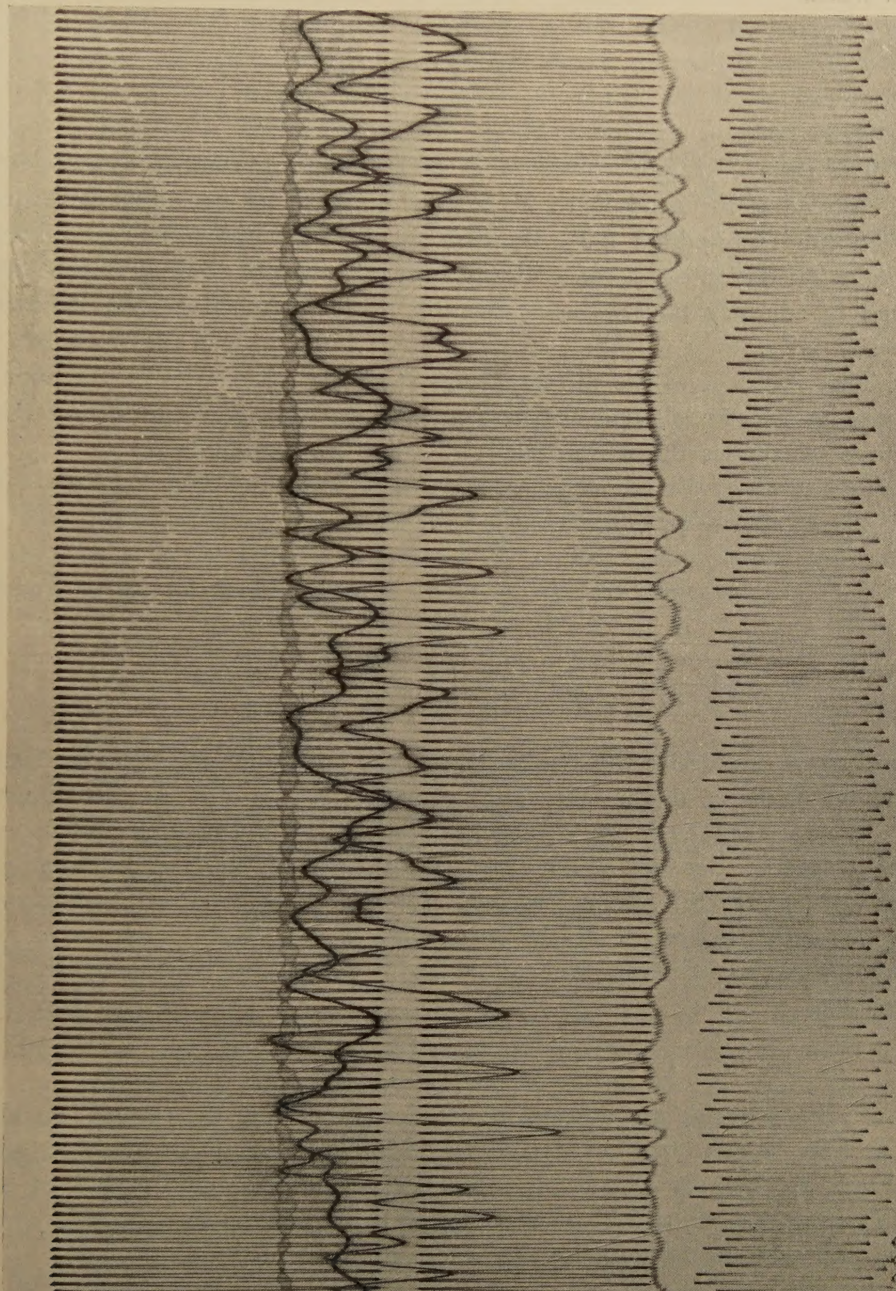

Fig. 2.

Record at a depth of 17 m (keel at 20 m); course slowly changing from N 150° E (right) to N 140° E (left); principal wave-period 7 sec.; wave-length according to Rankine 76 m; principal wave coming according to records of long period pendulums from N 50° E; this wave-direction agreed with the direction estimated at the sea-surface.

The vertical accelerations again show a phase-difference of $\frac{1}{2}\pi$ from the horizontal accelerations.

Double amplitude of the wave, as deduced from the accelerations, about 40 cm; according to Rankine double amplitude of the wave at the surface about 170 cm; this wave has been noticed at the surface.

in which

$$p_s = + \frac{1}{\frac{T_s^2}{T_\omega^2} - 1} \cdot \frac{a}{g}.$$

The ratio T_s^2/T_w^2 is considerably greater than 1 when the period of the period of the pendulum $2\,T_s$ is, as it ought to be, considerably larger than the wave-period $2\,T_w$. In that case the pendulum makes only a slight fluctuation compared with the apparent fluctuation of gravity $-a/g$ and in contrary phase. Superposed on this it can swing with its own period $2\,T_s$.

The record gives the difference $\theta_a - \theta_s$ of the movements of the apparatus in the gimbals and the long period pendulum. So the part of this curve which has the wave-period $2\,T_w$ has the amplitude

$$r_w = -p_a + p_s = F \cdot \frac{a}{g}$$

$$F = \frac{1}{1 - \frac{T_a^2}{T_\omega^2}} + \frac{1}{\frac{T_s^2}{T_\omega^2} - 1} = \frac{1 - \frac{T_a^2}{T_s^2}}{\left(1 - \frac{T_a}{T_s}\right)^2 - \left(\frac{T_\omega}{T_s} - \frac{T_a}{T_\omega}\right)^2}$$

and so we can derive the amplitude a of the horizontal accelerations from the amplitude r_w of the record; the periods T_a, T_w and T_s are all known.

We see that the factor F, which we might call the enlargement-factor of the record, depends of the period of the waves. This is a disadvantage as it causes a different enlargement for the different wave movements, which may be present at the same time. For a fairly large range of wave-periods, however, we can make F practically constant, viz. by making

$$T_a \cdot T_s = T_w^2 \text{ (T_w the mean value of the range).}$$

In that case the second term of the denominator of the second formula for F is zero for the mean wave-period and it is small for wave-periods which are near to this mean value. This condition can be realized by adjusting the period $2\,T_s$ of the long period pendulum in a suitable way. In this case the formula for F reduces to

$$F = \frac{T_s + T_a}{T_s - T_a}.$$

The result obtained provisionally from the records of both trips, that, at least approximately, the resultant of the two components $\ddot{y}$ and $\ddot{z}$ of the horizontal acceleration is equal to the vertical acceleration $\ddot{x}$ (with a phase-difference of $\frac{1}{2}\pi$) is highly important for the question whether

it is still possible to apply the correction for the BROWNE terms to the results of the old gravity determinations at sea; from the records of these observations the vertical accelerations can be derived. It is fortunate in this regard that it thus seems possible to derive the combined effect of the three terms from the vertical acceleration only, without needing data about the direction of the wave-movement. This last point is also fortunate because the records show that in many cases there are several wave-movements from different sides at the same time and the estimate at the sea surface of the direction of the main wave-movement is not always true for the depth of submergence.

So we put

$$\delta g = -\frac{\overline{\ddot{x}^2}}{4g} + \frac{\overline{\ddot{y}^2}}{2g} + \frac{\overline{\ddot{z}^2}}{2g} = + \frac{\overline{\ddot{x}^2}}{4g}$$

in which the dashes indicate mean values during the whole duration.

The mean value of this combined effect can be derived from the fluctuations of the chronometer-marks in the main pendulum-records by means of the formula on page 4 (651) of the first paper

$$\delta g = \frac{g}{2}\left(\frac{T\alpha}{T_w a}\right)^2$$

T = pendulum period,
a = pendulum amplitude on the record,
T_w = half-period of the wave movement,
α = amplitude of the fluctuations of the chronometer-marks at the centre of the pendulum-record.

This formula has been applied for all the old pendulum records. In the middle half of one of the two main pendulum records the fluctuations were measured of one of the chronometer-curves, i.e. as far as it was comprised between two lines on both sides parallel to the axis of the record and halfway between the axis and the edges of the record. The mean value of these fluctuations was divided by the amplitude at that place of the record and the square of this ratio was taken. This was done for ten different chronometer-curves, chosen at random in the record, and the mean of these squares was introduced in the formula as the value of $(\alpha/a)^2$. For the computation of T_w the number of wave-lengths was counted in one minute, estimating also fractions when it was not a whole number, and this was done for the same ten chronometer-curves; from the mean value of these numbers T_w was derived. The whole formula was multiplied by 1.097 in order to take into account that the fluctuation of the chronometer-curve is to be measured in the middle of the record and that in measuring a fluctuation at some distance from the middle, we find a value that is multiplied with *cos* φ if φ is the phase difference from the middle of the record. In measuring in the above way we get a mean value of

Second order Corrections for Ship's Movements (BROWNE's *Terms*)
for Gravity Observations at Sea, 1923—1932 (in mgal)

33 — 18	80 — 5	123 — 4	201 — 14	371 — 3
34 — 18	81 — 6	124 — 4	202 — 9	372 — 2
36 — 4	82 — 8	125 — ?	203 — 5	373 — 4
37 — 26	83 — 5	126 — 6	204 — 3	374 — 5
38 — 11	84 — 4	128 — 3	205 — 4	376 — 4
39 — 8	85 — 6	129 — 2	206 — 4	377 — 6
40 — 23	86 — 6	130 — 2	207 — 5	378 — 7
41 — 6	87 — 6	135 — 5	208 — 1	379 — 2
42 — 2	88 — 3	137 — 4	209 — 2	380 — 4
43 — 8	89 — 6	138 — 5	210 — 2	381 — 5
44 — 2	90 — 15	140 — 5	212 — 2	388 — 2
45 — 22	91 — 2	141 — 5	213 — 5	389 — 2
46 — 9	92 — 9	142 — 5	214 — 4	390 — 3
48 — 4	93 — 5	145 — 2	215 — 8	391 — 3
49 — 4	94 — 10	170 — 7	216 — 8	392 — 3
50 — 4	95 — 5	171 — 4	217 — 5	397 — 4
51 — 6	96 — 5	172 — 6	218 — 7	398 — 6
52 — 10	98 — 7	173 — 5	219 — 9	426 — 9
54 — 6	99 — 6	174 — 9	220 — 9	427 — 3
55 — 8	100 — 10	176 — 3	225 — 13	428 — 3
56 — 4	101 — 5	177 — 3	226 — 11	430 — 2
57 — 44	102 — 4	178 — 2	227 — 10	431 — 3
58 — 9	103 — 3	179 — 4	228 — 6	441 — 3
59 — 9	104 — 3	180 — 2	239 — 7	442 — 3
60 — 8	105 — 8	181 — 3	242 — 5	443 — 1
61 — 11	106 — 4	182 — 4	245 — 4	444 — 1
62 — 5	107 — 6	183 — 5	246 — 7	458 — 2
63 — 4	108 — 3	184 — 6	257 — 6	459 — 4
64 — 5	109 — 8	185 — 2	258 — 2	460 — 2
65 — 8	110 — 4	186 — 4	267 — 6	470 — 3
66 — 4	112 — 4	187 — 5	268 — 4	471 — 2
67 — 3	114 — 4	188 — 4	269 — 6	472 — 2
68 — 3	115 — 3	189 — 5	279 — 7	482 — 2
69 — 3	116 — 3	190 — 2	280 — 7	
70 — 5	117 — 4	191 — 3	281 — 6	
73 — 4	118 — 4	192 — 2	290 — 9	
74 — 4	118 — 4	192 — 2	290 — 9	
75 — 12	120 — 5	197 — 9	294 — 18	
78 — 5	121 — 3	199 — 11	369 — 3	
79 — 8	122 — 3	200 — 7	370 — 5	

The numbers of the stations refer to the list of stations in "Gravity Expeditions at Sea", 1923—1932, Vol. I, p. 101 e.s. and Vol. II, p. 89 e.s. The corrections for the stations 1—32 could not be determined but they are probably small as the sea was quiet during all the observations of this voyage. For station 183, the record was missing and so the correction could not be measured; for this station the mean was taken of Nos. 182 and 184, which were observed at short time-intervals before and after N°. 183. For the stations not mentioned the corrections are — 1 mgal or smaller.

the fluctuation which is multiplied with the mean value of $\cos\varphi$, φ ranging from —30° to +30°, i.e. with $3/\pi$ and so we have to multiply the formula for δg with the square of the reciprocal quantity, i.e. with $\frac{1}{9}\pi^2$.

For those records where the amplitude of the fluctuation did not exceed 0.6 mm only four chronometer-curves were measured for obtaining the mean value; for those where the amplitude of the fluctuation was smaller than 0.25 mm the correction, amounting in this case to about 0.8 milligal, was neglected.

The following list gives the corrections in milligal for the old gravity stations at sea of which the results have been published by the Netherlands Geodetic Commission in "Gravity Expeditions at Sea", 1923—1932, Vol. I, p. 101 e.s. and Vol. II, p. 89 e.s. Of the 486 stations the first 32 stations could not be corrected, because the records of these stations, observed with the old Stückrath pendulum apparatus, practically do not allow their computation. The corrections of the other stations that are not mentioned are 1 milligal or less.

The mean error of these corrections as far as they result from the errors of measuring is small, but there are two other possible causes of error. In the first place the assumption that had to be made about the relation of the horizontal and the vertical accelerations of the ship; in the future when more is known about these movements it will be better possible than it is now to make an estimate of the degree of accuracy of this assumption. In the second place the formula that has been used to derive δg from the fluctuations in the chronometer-curves of the records, is based on the assumption of a pure harmonic fluctuation and in many cases the movement has been more complicated. It is difficult to make an estimate now of the effect of these two causes of error but the writer thinks that in estimating the mean error of the corrections at one half of their value, a liberal value is obtained that is not likely to be too small.

In publishing these results, the writer wishes to express his sincere thanks to the Netherlands Navy and its representatives for the opportunities given for the necessary researches and to Mr. BROWNE for his kind cooperation in discussing the elucidation of these problems.

Medicine. — *Some remarks on vestibular nystagmus.* By A. DE KLEYN.

(Communicated at the meeting of May 28, 1938.)

For the experimental study of vestibular nystagmus we can best make use of the method of TOPOLANSKY-BARTELS. By this method the experiments are usually made on rabbits. As a general anaesthetie ether is administered; tracheotomy performed, respiration maintained artificially. The external and internal recti muscles of one eye are then isolated and the free ends connected by a thread to a lever, which registers the contractions of these muscles on a kymograph. After this the eyeball is extirpated and the skullvault and cerebrum removed.

As a result of a large series of experimental studies it has been ascertained that a normal vestibular nystagmus with slow and quick component can still be elicited:

a. after removal of the hemispheres.

b. after removal of the cerebellum.

c. after section through the brain just anterior to the abducens nuclei; the nuclei of the oculomotor and trochlear nerves being put out of function.

d. after section of the medulla, just behind the vestibular nuclei.

Thus the following small reflex arc is sufficient for a vestibular nystagmus with slow and quick phase:

peripheral labyrinth — vestibular nerve — vestibular nuclear area; abducens nucleus, abducens nerve and external rectus muscle [1]).

It is very remarkable that if only this arc is present, a nystagmus can be elicited in both directions i.e. not only a nystagmus consisting of slow contractions and quick relaxations of the external rectus, but likewise one with slow relaxations and quick contractions of the same muscle.

By these experiments on animals, however, the clinicians were not yet convinced at all. Again and again one could read that this might perhaps be true for the rabbit, but that man was no rabbit and that various facts indicated that the quick phase in man had a cortical origin. In favour of this view was the observation that on vestibular stimulation the quick phase may be absent e.g. in some forms of idiocy, in epileptic insults, etc. so that only a deviation of the eyes ensues. This conception is still often found in clinical literature and sometimes even in physiological papers. A priori however it is not very probable that in such a primordial reflex as vestibular nystagmus is, fundemental differences should be found in man and animals.

By a quite fortuitous finding it could in fact be proved, that in man

[1]) A. DE KLEYN, Proc. Kon. Akad. v. Wetensch. **23**, (1921), 1. Arch. f. Ophthalmologie 107, 480, (1922).

as well the same small reflex arc as in the rabbit is sufficient for a complete nystagmus. In 1929 in the Maternity Department at Utrecht an anencephalous monstrum was born that survived still for a week and could be accurately studied during that time [1]). In this monstrum on stimulating the vestibular organs a nystagmus with slow and quick phase could be elicited in both directions. After its death it was found that the monstrum:

a. had no hemispheres.

b. had no cerebellum.

c. had no midbrain, so that the oculomotor and trochlear nuclei were absent.

d. all ocular muscles were absent except both external muscles.

On the contrary the peripheral labyrinth, the vestibular nerves, abducens nuclei and abducens nerves had a normal development. A more close accordance with the experimental findings surely cannot be imagined, and besides those events show, that experiments, carried out with a scientific purpose only, may have clinical application.

Although for the arousing of vestibular nystagmus only a small reflex arc, comprising a very small area of the central nervous system, is required, this in no way means, that in actual life in the human or animal organism this reflex arc is always used unaffected by other influences and so a given stimulus of the vestibular organ will always be followed by the same vestibular nystagmus.

It must be clear, that in varying degrees of irritability of the peripheral labyrinth the nystagmus will likewise vary. The condition of the nuclear area of the vestibular organ and the ocular muscles is also of much importance. By their multiple connections with other parts of the central nervous system these can be under the influence of other stimuli to an extent varying from one moment to another, and thus can have a varying influence on the reflex arc of vestibular nystagmus [2]). In the same way, it will not be indifferent in which condition the effectory organ, the ocular muscle, finds itself. In order to investigate this with my friend LE HEUX, the following experiment was carried out. During a vestibular nystagmus which was being recorded by the TOPOLANSKY-BARTELS method, a solution of eatylcholine or nicotine was applicated on the orbital contents in which the ocular muscles had been dissected free. As a result in both cases we found a distinct retardation of the frequency of the nystagmus. The peculiar feature however was that not only the eye with the dissected eye muscles showed this diminuated frequency, but that also the other intact eye followed synchronically with the same slowing down of frequency. Now one might think that the aetylcholine and nicotine are quickly absorbed

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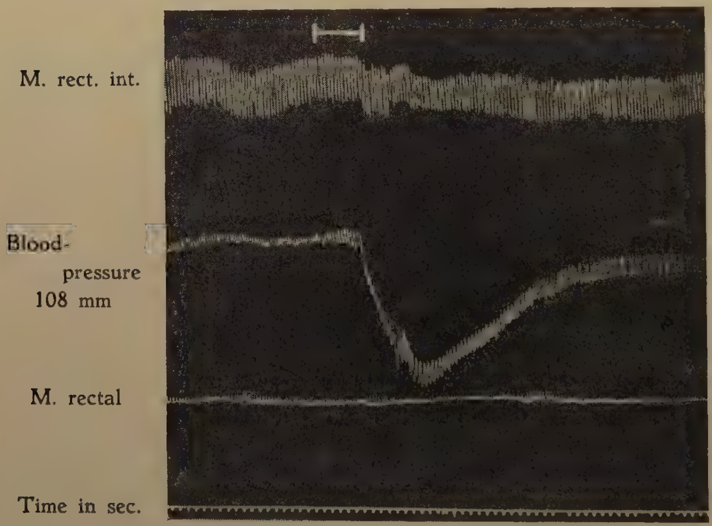
[1]) A. DE KLEYN und V. SCHENK, Acta-oto-laryngologica **15**, 439 (1931).

[2]) S.i.a. A. DE KLEYN and C. VERSTEEGH, Acta oto-laryngologica **6**, 38 (1923). J. G. DUSSER DE BARENNE und A. DE KLEYN, Arch. f. Ophthalmologie 111, 374 (1923). J. LE HEUX und A. DE KLEYN, Proc. Kon. Akad. v. Wetensch., Amsterdam, **38**, 3 (1935).

from the orbita and the decrease of frequency could be explained by a general action on the central nervous system. That this cannot be true is shown by the following experiments:

↑ Acetylcholine 0.2 % in the orbita. Caloric-stimulation of the left can with cold water Vth nerve severed-intracranially.

Fig. 1. On instillation of acetylcholine in the orbita a distinct lowering of the nystagmus-frequency follows. Absorption of the drag is in any case so slight, that no influence can be observed on the blood pressure.



Thalamus rabbit. |—| Infection of cholin intraveneously. (3.5 mgrm pr. kgrm. Experiment of Dr. MEURMAN.

Fig. 2. Choline injected intraveneously distinctly acts on the blood pressure, but does not bring about any marked change in the nystagmus. So it is clear, that concerning a vestibular nystagmus it is really not indifferent in which condition the effectory organ, the ocular muscle, finds itself.

The second subject I propose to deal with concerns the so-called successive induction which can be studied in vestibular nystagmus in a simple way, as was shown some time ago [1]).

When a rabbit submitted to alcohol intoxication is placed in lateral decubitus on one or other side, a vestibular nystagmus appears. In this condition of poisoning namely no nystagmus occurs so long as the head remains in normal position in space, but it appears as soon as it comes in a lateral position. We are dealing here with a positional nystagmus i.e.a. nystagmus depending on the position of the head in space. This is a vestibular form of nystagmus because if in a rabbit both labyrinths are removed, no nystagmus can be provoked in alcohol poisoning; other facts likewise indicate, that this nystagmus is elicited in the labyrinths.

The following experiment may serve as an illustration.

A rabbit submitted to alcohol intoxication was placed on its left side. The nystagmus began ten minutes after the intoxication and steadily increased in frequency till a maximum of 100 beats a minute was reached after 30 minutes. It disappeared after one hour. When thereupon the animal was placed in the right decubitus, again a strong spontaneous nystagmus appeared; which lasted 20 minutes. Now the animal again was turned into left side decubitus, in which position the nystagmus had completely disappeared before and none the less the nystagmus reappeared.

Probably in this case we have to deal with successive induction, which was described by SHERRINGTON in spinal reflexes. The reflex-arc of one of the vestibular systems during its own activity induces in the antagonistic arc a phase of greater excitability and capacity for discharge.

During the last months we found, that in patients this induction can also be demonstrated.

In 10 cases the following phenomenon was observed. When the patients were looking straight forward no nystagmus was present; by looking to the right a nystagmus with the quick component to the right, by looking to the left a nystagmus with the quick component to the left appeared. However when at first the patient was looking to the right during a longer time (1-2 minutes) and then looked straight forward a nystagmus with the quick component to the left appeared, whereas in normal circumstances this regard was not accompanied by a mystagmus.

On the other hand when the patient was looking during a longer time to the left and then looked straight forward a nystagmus with the quick component to the right appeared.

In this way in consequence of the above mentioned successive induction it was possible to arrange that the patient by looking straight forward either showed no nystagmus at all or a nystagmus to the right or a nystagmus to the left.

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[1]) LE HEUX und A. DE KLEYN, Proc. Kon. Akad. v. Wetensch., Amsterdam, **40**, 326 (1937).

Mathematics. — *Contribution à la théorie additive des nombres.* Par J. G. VAN DER CORPUT. (Quatrième communication [1]).

(Communicated at the meeting of May 28, 1938.)

Lemme 13: *Si q et U désignent des nombres naturels, $\chi(x)$ un polynôme à coefficients entiers, et si l'on pose pour toute fraction irréductible* $\frac{a}{q}$

$$\lambda\left(\frac{a}{q}\right)=\frac{\varphi(U)}{\varphi(qU)}\sum_{\substack{h=1\\ h\equiv u\,(mod.\,U)\\ (h,q)=1}}^{qU} e^{\frac{2\pi i a\chi(h)}{q}},$$

on a pour chaque paire de fractions irréductibles $\frac{a_1}{q_1}$ *et* $\frac{a_2}{q_2}$ *dont les dénominateurs q_1 et q_2 sont premiers entre eux*

$$\lambda\left(\frac{a_1}{q_1}\right)\lambda\left(\frac{a_2}{q_2}\right)=\lambda\left(\frac{a_1}{q_1}+\frac{a_2}{q_2}\right).$$

Démonstration: On a

$$\lambda\left(\frac{a_1}{q_1}\right)\lambda\left(\frac{a_2}{q_2}\right)=\frac{\varphi^2(U)}{\varphi(q_1U)\,\varphi(q_2U)}\sum_{\substack{h_1=1\\ h_1\equiv u_1\\ (h_1,q_1)=1}}^{q_1U}\ \sum_{\substack{h_2=1\\ h_2\equiv u_2\,(mod.\,U)\\ (h_2,q_2)=1}}^{q_2U} e^{2\pi i\left(\frac{a_1}{q_1}\chi(h_1)+\frac{a_2}{q_2}\chi(h_2)\right)}.$$

A toute paire d'entiers k_1 et k_2 tels que

$$0<h_1=u+k_1U\leqq q_1U \text{ et } 0<h_2=u+k_2U\leqq q_2U$$

correspond un seul entier k tels que l'on ait

$$0<h=u+kU\leqq q_1q_2U,$$

$$k\equiv k_1 \ (mod.\ q_1) \text{ et } k\equiv k_2 \ (mod.\ q_2).$$

Alors on a

$$h\equiv h_1 \ (mod.\ q_1) \text{ et } h\equiv h_2 \ (mod.\ q_2).$$

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[1]) Dans la première communication (p. 228) j'ai dit que presque tout multiple de 24 augmenté de 3 est la somme des carrés de trois nombres premiers. Il va sans dire qu'on doit faire une exception pour les multiples de 5. Dans mon article: „Sur deux, trois ou quatre nombres premiers", Troisième communication, p. 103, il faut supposer que n soit égal à une puissance p^λ d'un nombre premier p, et on doit poser $\varepsilon=(-1)^\lambda$ (ou lieu de $\varepsilon=-1$) dans le cas particulier où n est congru à -1 (*mod.* 4) et a est un non-reste de n.

Si h_1 est premier avec q_1 et en même temps h_2 avec q_2, le nombre h est premier avec $q_1 q_2$ et inversement. Ainsi on obtient

$$\lambda\left(\frac{a_1}{q_1}\right)\lambda\left(\frac{a_2}{q_2}\right)=\frac{\varphi^2(U)}{\varphi(q_1 U)\,\varphi(q_2 U)}\sum_{\substack{h=1\\ h\equiv u\,(mod.\,U)\\ (h,q_1 q_2)=1}}^{q_1 q_2 U} e^{2\pi i\left(\frac{a_1}{q_1}+\frac{a_2}{q_2}\right)\chi(h)}$$

$$=\frac{\varphi(U)}{\varphi(q_1 q_2 U)}\sum_{\substack{h=1\\ h\equiv u\,(mod.\,U)\\ (h,q_1 q_2)=1}}^{q_1 q_2 U} e^{2\pi i\left(\frac{a_1}{q_1}+\frac{a_2}{q_2}\right)\chi(h)},$$

parce que la relation

$$\prod_{p\mid q_1 U}\left(1-\frac{1}{p}\right).\prod_{p\mid q_2 U}\left(1-\frac{1}{p}\right)=\prod_{p\mid U}\left(1-\frac{1}{p}\right).\prod_{p\mid q_1 q_2 U}\left(1-\frac{1}{p}\right)$$

nous fournit la formule

$$\varphi(q_1 U)\,\varphi(q_2 U)=\varphi(U)\,\varphi(q_1 q_2 U).$$

De cette manière le lemme 13 est démontré.

Je vais démontrer maintenant la proposition 9. Dans cette démonstration $c_{87}, c_{88}, \dots, c_{109}$ dépendent uniquement de M, ν, K, U et du choix du polynôme $\psi(x)$.

Première partie de la démonstration:

Dans cette partie j'introduis quelques intervalles et quelques fonctions vérifiant les conditions de la proposition 1.

1. Je choisis pour V l'intervalle fermé $(2K, N)$, pour V' l'intervalle fermé (A', N) où $A'=N n^{-\frac{1}{2}gM}$ et pour T l'intervalle fermé $(2, N)$; comme on le sait, n est égal à $\log N$. Chacun de ces intervalles renferme moins de N entiers et, si N est suffisamment grand, au moins un entier.

Posons $r(v)=1$ ou 0, selon que $\frac{v}{K}$ est un nombre premier ou non; de cette définition il suit

$$\sum_{v}|r(v)|^2=\sum_{p\leqq K^{-1}N}1<N.$$

Si nous prenons

$$\varrho(v)=\frac{1}{K\,\varphi(U)\log\frac{v}{K}}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (37)$$

nous avons

$$\sum_{v}|\varrho(v)|^2=\frac{1}{K^2\varphi^2(U)}\sum_{2K\leqq v\leqq N}\frac{1}{\log^2\frac{v}{K}}\leqq\frac{N}{\log^2 2},$$

de sorte que les inégalités (1) sont vérifiées si l'on prend $\Gamma=\frac{1}{\log 2}$.

Pour un entier v' auquel correspond au moins un nombre naturel x tel que $\psi(x) = v'$ je pose $r'(v') = 1$; pour les autres v' je pose $r'(v') = 0$. Par conséquent

$$\sum_{v'} |r'(v')| = \sum_{\substack{A' \leqq v' \leqq N \\ v' = \psi(x)}} 1 \leqq c_{87} N^{\frac{1}{g}} \leqq c_{87} N A'^{-1+\frac{1}{g}}.$$

Je prendrai

$$\Gamma' = c_{88} A'^{-1+\frac{1}{g}} \quad \ldots \ldots \ldots \ldots \quad (38)$$

La première des inégalités (2) est donc vérifiée dès que l'on choisit $c_{88} \geqq c_{87}$.

Les nombres

$$\varrho'(v') = g^{-1} b^{-\frac{1}{g}} v'^{-1+\frac{1}{g}} \qquad (A' \leqq v' \leqq N) \quad \ldots \quad (39)$$

et

$$\sum_{v' \text{ et } v'+1 \text{ dans } V'} |\varrho'(v'+1) - \varrho'(v')|$$

sont $\leqq g^{-1} b^{-\frac{1}{g}} A'^{-1+\frac{1}{g}}$; la première condition de la proposition 1 est donc vérifiée si l'on impose en outre à c_{88} la condition $c_{88} \geqq g^{-1} b^{-\frac{1}{g}}$.

2. Prenons $l = 1$ et en outre pour toute fraction irréductible

$$\lambda\left(\frac{a}{q}\right) = \frac{\varphi(U)}{\varphi(qU)} \sum_{\substack{h=1 \\ h \equiv u (\mathrm{mod.}\, U) \\ (h,q)=1}}^{qU} e^{\frac{2\pi i a K h}{q}} \quad \ldots \ldots \quad (40)$$

et

$$\lambda'\left(\frac{a}{q}\right) = \frac{1}{q} \sum_{h=1}^{q} e^{\frac{2\pi i a \psi(h)}{q}} \quad \ldots \ldots \ldots \quad (41)$$

$\lambda\left(\frac{a}{q}\right)$ et $\lambda'\left(\frac{a}{q}\right)$ sont en valeur absolue $\leqq q$, de sorte que l'inégalité (3) est vérifiée, si l'on choisit $\gamma_1 \geqq 1$.

On a pour tout nombre réel y

$$\sum_{v \leqq y} r(v) e^{\frac{2\pi i a v}{q}} - \lambda\left(\frac{a}{q}\right) \sum_{v \leqq y} \varrho(v)$$

$$= \sum_{\substack{p | qU \\ p \leqq K^{-1} y \\ p \leqq K^{-1} N}} e^{\frac{2\pi i a K p}{q}} + \sum_{\substack{h=1 \\ h \equiv u (\mathrm{mod.}\, U) \\ (h, qU)=1}}^{qU} e^{\frac{2\pi i a K h}{q}} \left\{ \sum_{\substack{p \leqq K^{-1} y \\ p \leqq K^{-1} N \\ p \equiv h (\mathrm{mod.}\, qU)}} 1 - \frac{1}{\varphi(qU)} \sum_{\substack{v \leqq y \\ 2 \leqq v \leqq N}} \frac{1}{K \log \frac{v}{K}} \right\}$$

et la valeur absolue du dernier membre est d'après le lemme 8 (appliqué avec $k = qU$) pour tout nombre naturel m inférieur à

$$qU + q C_1 N n^{-m} < C_2 N q n^{-m};$$

dans cette partie de la démonstration $C_1, C_2, \ldots, C_6$ dépendent uniquement de m, M, K, U et du choix du polynôme $\psi(x)$.

Le lemme 9 nous apprend que

$$\sum_{v' \leqq y} r'(v') e^{\frac{2\pi i a v'}{q}} - \lambda'\left(\frac{a}{q}\right) \sum_{v' \leqq y} \varrho'(v')$$

$$= \sum_{h=1}^{q} e^{\frac{2\pi i a \psi(h)}{q}} \left\{ \sum_{v' \leqq y}^{\star} 1 - q^{-1} g^{-1} b^{-\frac{1}{g}} \sum_{v' \leqq y} v'^{-1+\frac{1}{g}} \right\}$$

est en valeur absolue

$$\leqq c_{89}\, q \leqq C_3\, q\, A'^{-1+\frac{1}{g}} N n^{-m}$$

pour tout nombre réel y et pour tout nombre naturel m. Par conséquent les inégalités (4) et (5) sont vérifiées, si l'on prend

$$\gamma_m \geqq C_2 \text{ et } \geqq C_3; \qquad c_{88} \geqq 1.$$

3. Il existe un nombre c_{90} tel que $\psi'(x)$ soit constamment positif ou constamment négatif pour $x \geqq c_{90}$. On a pour toute valeur réelle de α

$$\Big| \sum_{v' \leqq c_{90}} r'(v')\, e^{2\pi i \alpha v'} \Big| \leqq c_{91}$$

et

$$\sum_{v' > c_{90}} r'(v')\, e^{2\pi i \alpha v'} = \sum_{x}{}_1\, e^{2\pi i \alpha \psi(x)},$$

x parcourant un système de nombres naturels consécutifs. Je démontrerai maintenant qu'à tout nombre naturel m correspond un entier η_m dépendant uniquement de m et du choix du polynôme $\psi(x)$ jouissant de la propriété suivante: pour tout nombre réel α tel que l'intervalle $(\alpha \mp N^{-1} n^{\eta_m})$ ne contienne aucune fraction à dénominateur positif $\leqq n^{\eta_m}$, on a

$$\Big| \sum_{v'} r'(v')\, e^{2\pi i \alpha v'} \Big| \leqq C_4 N^{\frac{1}{g}} n^{-m} \leqq C_4 N A'^{-1+\frac{1}{g}} n^{-m}. \qquad (42)$$

Nous pouvons supposer que le nombre Z des termes de la somme $\sum_{x}{}_1$ soit supérieur à $N^{\frac{1}{g}} n^{-m}$, la relation (42) étant manifeste dans l'autre cas. On a $Z \leqq c_{92} N^{\frac{1}{g}}$. Le lemme 11 nous fournit un nombre ζ dépendant uniquement de m et du choix du polynôme $\psi(x)$ et jouissant de la propriété mentionnée dans ce lemme 11. Si nous posons $\eta_m = 1 + \zeta + gm$, nous avons

$$N^{-1} n^{\eta_m} \geqq Z^{-g} (\log Z)^{\zeta} \text{ et } n^{\eta_m} \geqq (\log Z)^{\zeta},$$

supposé que N soit suffisamment grand (sinon, (42) est évident). Si l'intervalle $(\alpha \mp N^{-1} n^{\eta_m})$ ne contient aucune fraction à dénominateur positif $\leqq n^{\eta_m}$, l'intervalle $(\alpha \mp Z^{-g} (\log Z)^{\zeta})$ ne contient aucune fraction

à dénominateur positif $\leqq (\log Z)^{\tau}$, et le lemme 11 nous apprend pour tout nombre naturel m

$$\left| \sum_{x} {}_1\, e^{2\pi i \alpha \psi(x)} \right| \leqq C_5 Z (\log Z)^{-m} \leqq C_6 N^{\frac{1}{g}} n^{-m}.$$

Ainsi nous trouvons (42), d'où il suit que les conditions de la proposition 1 sont vérifiées si l'on choisit $\gamma_m \geqq C_6$ et $c_{88} \geqq 1$.

Deuxième partie de la démonstration:

Ayant démontré dans la première partie que les conditions de la proposition 1 sont valables, je vais considérer maintenant celles de la proposition 5. Pour démontrer que la fonction

$$H(q, t) = \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} \lambda\left(\frac{a}{q}\right) \lambda'\left(\frac{a}{q}\right) e^{-\frac{2\pi i a t}{q}}$$

possède, pour toute paire de nombres naturels q_1 et q_2 qui sont premiers entre eux, la propriété multiplicative exprimée par (18), il suffit d'étudier les relations (19) et (20), $\frac{a_1}{q_1}$ et $\frac{a_2}{q_2}$ désignant deux fractions irréductibles dont les dénominateurs q_1 et q_2 sont premiers entre eux. Le lemme 13, appliqué avec $\chi(x) = Kx$ nous apprend que la fonction $\lambda\left(\frac{a}{q}\right)$, définie par (40), satisfait à (19). Pour la fonction $\lambda'\left(\frac{a}{q}\right)$, définie par (41), on a

$$\lambda'\left(\frac{a_1}{q_1}\right) \lambda'\left(\frac{a_2}{q_2}\right) = q_1^{-1} q_2^{-1} \sum_{h_1=1}^{q_1} \sum_{h_2=1}^{q_2} e^{2\pi i \left(\frac{a_1}{q_1} \psi(h_1) + \frac{a_2}{q_2} \psi(h_2)\right)};$$

le nombre h, déterminé par

$$h \equiv h_1 \,(\mathrm{mod.}\, q_1); \qquad h \equiv h_2 \,(\mathrm{mod.}\, q_2); \qquad 0 < h \leqq q_1 q_2$$

parcourt le système des nombres naturels $\leqq q_1 q_2$; on a donc

$$\lambda'\left(\frac{a_1}{q_1}\right) \lambda'\left(\frac{a_2}{q_2}\right) = q_1^{-1} q_2^{-1} \sum_{h=1}^{q_1 q_2} e^{2\pi i \left(\frac{a_1}{q_1} + \frac{a_2}{q_2}\right) \psi(h)},$$

d'où suit (20). $H(q, t)$ possède donc la susdite propriété multiplicative.

Evaluons maintenant

$$\lambda\left(\frac{a}{p^{\sigma}}\right) = \frac{\varphi(U)}{\varphi(p^{\sigma} U)} \sum_{\substack{h=1 \\ h \equiv u\,(\mathrm{mod.}\, U) \\ (h,p)=1}}^{p^{\sigma} U} e^{\frac{2\pi i a K h}{p^{\sigma}}}$$

pour un nombre premier quelconque p et pour un nombre naturel quelconque σ. Dans le cas où p n'est pas un facteur de U, nous avons

$$\lambda\left(\frac{a}{p^\sigma}\right)=\frac{1}{p^{\sigma-1}(p-1)}\left\{\sum_{\substack{h=1\\ h\equiv u\,(\mathrm{mod.}\,U)}}^{p^\sigma U} e^{\frac{2\pi i a K h}{p^\sigma}}-\sum_{\substack{h=1\\ h\equiv u\,(\mathrm{mod.}\,U)\\ h\equiv 0\,(\mathrm{mod.}\,p)}}^{p^\sigma U} e^{\frac{2\pi i a K h}{p^\sigma}}\right\}$$

$$=\frac{1}{p^{\sigma-1}(p-1)}(p^\sigma-p^{\sigma-1})=1 \text{ lorsque } p^\sigma/K; \quad . \quad . \quad . \quad . \quad (43)$$

$$=\frac{-p^{\sigma-1}}{p^{\sigma-1}(p-1)}=\frac{-1}{p-1} \text{ lorsque } p^\sigma \dagger K \text{ et } p^{\sigma-1}/K; \quad . \quad (44)$$

$$=0 \qquad \text{lorsque } p^{\sigma-1}\dagger K. \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Si par contre p est un facteur de U, nous avons, parce que u est premier avec U,

$$\lambda\left(\frac{a}{p^\sigma}\right)=\frac{1}{p^\sigma}\sum_{\substack{h=1\\ h\equiv u\,(\mathrm{mod.}\,U)}}^{p^\sigma U} e^{\frac{2\pi i a K h}{p^\sigma}}=e^{\frac{2\pi i a K u}{p^\sigma}} \text{ ou } 0, \quad . \quad . \quad . \quad . \quad (46)$$

selon que KU est divisible par p^σ ou non.

Evaluons ensuite

$$H(q,t)=\frac{1}{q}\sum_{h=1}^{q}\sum_{\substack{a=0\\ (a,q)=1}}^{q-1}\lambda\left(\frac{a}{q}\right)e^{\frac{2\pi i a(\psi(h)-t)}{q}}$$

dans le cas particulier où $q=p^\sigma$. Dans ce calcul nous utiliserons les relations

$$\left.\begin{aligned}\sum_{\substack{a=0\\ (a,p)=1}}^{p^\sigma-1} e^{\frac{2\pi i a k}{p^\sigma}} &= \sum_{a=0}^{p^\sigma-1} e^{\frac{2\pi i a k}{p^\sigma}}-\sum_{a=0}^{p^{\sigma-1}-1} e^{\frac{2\pi i a k}{p^{\sigma-1}}}\\ &=p^\sigma-p^{\sigma-1} \text{ lorsque } p^\sigma/k;\\ &=-p^{\sigma-1} \quad \text{lorsque } p^\sigma\dagger k \text{ et } p^{\sigma-1}/k;\\ &=0 \qquad \text{lorsque } p^{\sigma-1}\dagger k.\end{aligned}\right\} \quad . \quad . \quad (47)$$

Distinguons cinq cas:

1. Soit $p \nmid U$ et p^σ / K. On a

$$\left.\begin{aligned}
H(p^\sigma, t) &= p^{-\sigma} \sum_{h=1}^{p^\sigma} \sum_{\substack{a=0 \\ (a,p)=1}}^{p^\sigma-1} e^{\frac{2\pi i a(\psi(h)-t)}{p^\sigma}} \\
&= p^{-\sigma} \sum_{\substack{h=1 \\ p^\sigma / \psi(h)-t}}^{p^\sigma} (p^\sigma - p^{\sigma-1}) - p^{-\sigma} \sum_{\substack{h=1 \\ p^\sigma \nmid \psi(h)-t \\ p^{\sigma-1} / \psi(h)-t}}^{p^\sigma} p^{\sigma-1} \\
&= \sum_{\substack{h=1 \\ p^\sigma / \psi(h)-t}}^{p^\sigma} 1 - p^{-1} \sum_{\substack{h=1 \\ p^{\sigma-1} / \psi(h)-t}}^{p^\sigma} 1 \\
&= \sum_{\substack{h=1 \\ p^\sigma / \psi(h)-t}}^{p^\sigma} 1 - \sum_{\substack{h=1 \\ p^{\sigma-1} / \psi(h)-t}}^{p^{\sigma-1}} 1
\end{aligned}\right\} \quad . \ . \quad (48)$$

2. Soit $p \nmid U$; $p^\sigma \nmid K$; $p^{\sigma-1} / K$. On a

$$\left.\begin{aligned}
-(p-1)\, H(p^\sigma, t) &= p^{-\sigma} \sum_{h=1}^{p^\sigma} \sum_{\substack{a=0 \\ (a,p)=1}}^{p^\sigma-1} e^{\frac{2\pi i a(\psi(h)-t)}{p^\sigma}} \\
&= \sum_{\substack{h=1 \\ p^\sigma / \psi(h)-t}}^{p^\sigma} 1 - \sum_{\substack{h=1 \\ p^{\sigma-1} / \psi(h)-t}}^{p^{\sigma-1}} 1.
\end{aligned}\right\} \quad . \ . \ . \quad (49)$$

3. Soit $p \nmid U$ et $p^{\sigma-1} \nmid K$. Dans ce cas $\lambda\left(\frac{a}{p^\sigma}\right)$ et $H(p^\sigma, t)$ s'annulent.

4. Soit p / U et p^σ / KU. On a

$$\left.\begin{aligned}
H(p^\sigma, t) &= p^{-\sigma} \sum_{h=1}^{p^\sigma} \sum_{\substack{a=0 \\ (a,p)=1}}^{p^\sigma-1} e^{\frac{2\pi i a(\psi(h)+Ku-t)}{p^\sigma}} \\
&= \sum_{\substack{h=1 \\ p^\sigma / \psi(h)+Ku-t}}^{p^\sigma} 1 - \sum_{\substack{h=1 \\ p^{\sigma-1} / \psi(h)+Ku-t}}^{p^{\sigma-1}} 1.
\end{aligned}\right\} \quad . \ . \ . \ . \quad (50)$$

5. Dans le cas où p / U et $p^\sigma \nmid KU$ les nombres $\lambda\left(\frac{a}{p^\sigma}\right)$ et $H(p^\sigma, t)$ s'annulent.

Si le nombre premier p n'est pas un facteur de KUG, il n'est pas un facteur du plus grand commun diviseur G des nombres $\psi(x) - \psi(0)$, où x est entier; les coefficients du polynôme $\psi(x) - \psi(0)$ ne sont donc pas tous divisibles par p, de sorte que la congruence

$$\psi(h) - t \equiv 0 \qquad (mod.\ p)$$

possède tout au plus g solutions et le nombre

$$-(p-1)\,H(p,t) = -1 + \sum_{\substack{h=1 \\ p \mid \psi(h)-t}}^{p^\sigma} 1$$

est $\geqq -1$ et $\leqq g-1$; par conséquent on obtient alors

$$|H(p,t)| \leqq \frac{g}{p-1} \leqq \frac{2g}{p} = \frac{2^{\frac{\log 2g}{\log 2}}}{p};$$

en outre on sait que $H(p^\sigma, t)$ s'annule pour tout entier $\sigma \geqq 2$ et tout nombre premier p qui n'est pas un facteur de KUG. Pour un nombre naturel q qui est divisible par le carré d'un nombre premier qui n'est pas un facteur de KUG, la fonction $H(q,t)$ s'annule donc en vertu de la propriété multiplicative de cette fonction; pour les autres nombres naturels q on a

$$|H(q,t)| \leqq c_{93} \prod_{\substack{p \mid q \\ p \nmid KUG}} \frac{2^{\frac{\log 2g}{\log 2}}}{p} \leqq \frac{c_{94}}{q} \left(\tau(q)\right)^{\frac{\log 2g}{\log 2}},$$

où $\tau(q)$ désigne le nombre des diviseurs de q. Ainsi on obtient pour tout nombre naturel q et pour tout entier t

$$|H(q,t)| \leqq \gamma_m \, q^{-1+\frac{1}{m}}$$

si γ_m est choisi supérieur à une borne convenable, dépendant uniquement de m, K, U, G et g.

Les conditions de la proposition 5 sont donc vérifiées.

Troisième partie de la démonstration:

La proposition 5, appliquée avec $m = gM$ (ν est donc $> m$), nous fournit l'inégalité

$$\sum_{t=2}^{[N]} |L(t) - A(t)\,\Omega_\nu(t)|^2 < c_{95}\, I'^2 N^3 n^{-gM},$$

d'où il suit en vertu de (38),

$$\sum_{t=2}^{[N]} |L(t) - A(t)\,\Omega_\nu(t)|^2 < c_{96}\, A'^{-2+\frac{2}{g}} N^3 n^{-gM} \quad . \quad . \quad . \quad (51)$$

Déduisons maintenant la relation

$$\Omega_\nu(t) = \Omega_\nu^*(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

pour tout entier $t > 1$ et tout ν tels que $(\log t)^\nu$ soit supérieur ou égal à $K^2 U^2$. Si p est un nombre premier et si p^ω désigne la puissance la plus élevée de p qui est un diviseur de KU, on a

$$1 + \sum_{\sigma=1}^{\infty} H(p^\sigma, t) = 1 + \sum_{\sigma=1}^{\omega+1} H(p^\sigma, t); \quad . \quad . \quad . \quad . \quad . \quad (53)$$

en effet, comme nous l'avons vu dans la seconde partie de cette démonstration, $H(p^\sigma, t)$ s'annule pour tout $\sigma > \omega + 1$. Dans le cas où p n'est pas un facteur de U, on a en vertu de (48)

$$1 + \sum_{\sigma=1}^{\omega} H(p^\sigma, t) = \sum_{\substack{h=1 \\ p^\omega | \psi(h) - t}}^{p^\omega} 1,$$

de sorte que (49) (appliqué avec $\sigma = \omega + 1$) nous fournit la formule

$$1 + \sum_{\sigma=1}^{\infty} H(p, t) = \sum_{\substack{h=1 \\ p^\omega | \psi(h) - t}}^{p^\omega} 1 - \frac{1}{p-1} \sum_{\substack{h=1 \\ p^{\omega+1} | \psi(h) - t}}^{p^{\omega+1}} 1 + \frac{1}{p-1} \sum_{\substack{h=1 \\ p^\omega | \psi(h) - t}}^{p^\omega} 1$$

$$= \frac{1}{p-1} \left\{ \sum_{\substack{h=1 \\ p^\omega | \psi(h) - t}}^{p^{\omega+1}} 1 - \sum_{\substack{h=1 \\ p^{\omega+1} | \psi(h) - t}}^{p^{\omega+1}} 1 \right\} = W(p, t),$$

d'après la définition de cette dernière fonction, donnée dans la troisième communication entre les propositions 7 et 8. Dans le cas où p est un facteur de U, le nombre $H(p^{\omega+1}, t)$ s'annule et on a en vertu de (53) et (50)

$$1 + \sum_{\sigma=1}^{\infty} H(p^\sigma, t) = \sum_{\substack{h=1 \\ p^\omega | \psi(h) + Ku - t}}^{p^\omega} 1 = W(p, t).$$

Pour tout entier $t > 1$ et tout ν tels que $(\log t)^\nu$ soit $\geqq K^2 U^2$ et pour tout nombre premier $p \leqq (\log t)^\nu$ on a

$$1 + \sum_{\sigma=1}^{\left[\frac{\nu \log\log t}{\log p}\right]} H(p^\sigma, t) = W(p, t).$$

En effet, si p est un facteur de KU, tout entier $\sigma > \left[\dfrac{\nu \log\log t}{\log p}\right]$ satisfait aux inégalités

$$p^\sigma > (\log t)^\nu \geqq K^2 U^2,$$

d'où il suit $\sigma > 2\omega \geqq \omega + 1$, de sorte que $H(p^\sigma, t)$ s'annule; si p n'est pas un facteur de KU, on a $\omega = 0$, de sorte que $H(p^\sigma, t)$ s'annule pour tout entier $\sigma > 1$, a fortiori pour tout entier $\sigma > \left[\dfrac{\nu \log\log t}{\log p}\right]$. Ainsi nous trouvons

$$\Omega_\nu(t) = \prod_p \Bigl(1 + \sum_{\sigma=1}^{\left[\frac{\nu \log\log t}{\log p}\right]} H(p^\sigma, t)\Bigr) = \prod_{p \leqq (\log t)^\nu} W(p, t) = \Omega_\nu^*(t)$$

d'après la définition de cette dernière fonction.

Nous pouvons donc remplacer dans (51) $\Omega_\nu(t)$ par $\Omega_\nu^*(t)$, si l'on remplace en même temps dans le second membre c_{96} par c_{97}; en effet, les contributions des termes dans lesquels $(\log t)^\nu$ est inférieur à $K^2 U^2$, sont inférieures à c_{98}.

Pour un nombre premier p qui n'est pas un facteur de KUG, le produit $(p-1)\,W(p, t)$ est d'après sa définition égal au nombre des nombres naturels $h \leqq p$ tels que $\psi(h) - t$ ne soit pas divisible par p. On a donc d'une part $(p-1)\,W(p, t) \leqq p$. Puisque p n'est pas un facteur de plus grand commun diviseur G de tous les nombres $\psi(x) - \psi(0)$, où x est entier, il est exclu que tous les coefficients du polynôme $\psi(x) - \psi(0)$ sont divisibles par p. Ainsi on obtient d'autre part $(p-1)\,W(p, t) \geqq p - g$. On a donc

$$|\,W(p, t) - 1\,| \leqq \frac{g}{p-1}, \quad \ldots\ldots\ldots \quad (53)$$

de sorte que le lemme 12 fournit l'inégalité

$$|\,\Omega_\nu^*(t)\,| \leqq c_{99}\,(\log\log t)^g \quad \ldots\ldots\ldots \quad (54)$$

On sait que

$$L(t) = \sum_{v+v'=t} r(v)\, r'(v')$$

est égal au nombre des nombres premiers $p \equiv u\,(mod.\ U)$ tels que t peut être mis sous la forme $t = Kp + \psi(x)$, où x est un nombre naturel avec $\psi(x) \geqq A'$. La différence $F(t) - L(t)$ est donc le nombre des nombres premiers $p \equiv u\,(mod.\ U)$ tels que $t - Kp$ est égal à un nombre de la forme $\psi(x) < A'$, c'est-à-dire

$$0 \leqq F(t) - L(t) \leqq c_{100}\, A'^{\frac{1}{g}}. \quad \ldots\ldots\ldots \quad (55)$$

En vertu de (37) et (39) on trouve

$$\Lambda(t) = \sum_{v+v'=t} \varrho(v)\,\varrho'(v') = c_{101} \sum_{\substack{v \geqq 2K \\ v' \geqq A' \\ v+v'=t}} \frac{v'^{-1+\frac{1}{g}}}{\log \frac{v}{K}},$$

où

$$c_{101} = \frac{1}{Kg\, b^{\frac{1}{g}}\, \varphi(U)},$$

par conséquent en vertu de (34)

$$0 \leqq \Phi(t) - \Lambda(t) = c_{101} \sum_{\substack{v \geqq 2K \\ 2 \leqq v' < A' \\ v+v'=t}} \frac{v'^{-1+\frac{1}{g}}}{\log \frac{v}{K}}.$$

Si t est $\geqq 2A'$, le dénominateur $\log\frac{v}{K}$ dans la dernière somme est supérieur à $\log\frac{t}{2K}$, par conséquent

$$\Phi(t) - \varLambda(t) < \frac{c_{102} A'^{\frac{1}{g}}}{\log t}.$$

Si $A' \leqq t < 2A'$, on a

$$\Phi(t) - \varLambda(t) \leqq c_{101} \sum_{\substack{v \geqq K\sqrt{A'} \\ 2 \leqq v' < A' \\ v+v'=t}} \frac{v'^{-1+\frac{1}{g}}}{\log\sqrt{A'}} + c_{101} \sum_{\substack{2K \leqq v < K\sqrt{A'} \\ 2 \leqq v' < A' \\ v+v'=t}} \frac{v'^{-1+\frac{1}{g}}}{\log 2}$$

$$< \frac{c_{103} A'^{\frac{1}{g}}}{\log A'} < \frac{c_{104} A'^{\frac{1}{g}}}{\log t}$$

et finalement dans le cas où $t < A'$ on obtient

$$\Phi(t) - \varLambda(t) \leqq c_{101} \sum_{\substack{v \geqq K\sqrt{t} \\ v' \geqq 2 \\ v+v'=t}} \frac{v'^{-1+\frac{1}{g}}}{\log\sqrt{t}} + c_{101} \sum_{\substack{2 \leqq v < K\sqrt{t} \\ v' \geqq 2 \\ v+v'=t}} \frac{v'^{-1+\frac{1}{g}}}{\log 2}$$

$$< \frac{c_{105} t^{\frac{1}{g}}}{\log t} < \frac{c_{105} A'^{\frac{1}{g}}}{\log t},$$

de sorte qu'on trouve dans tous les cas

$$0 \leqq \Phi(t) - \varLambda(t) < \frac{c_{106} A'^{\frac{1}{g}}}{\log t}. \quad \ldots \ldots \quad (56)$$

Les inégalités (54), (55) et (56) nous donnent maintenant le résultat

$$| F(t) - \Phi(t)\, \Omega_\nu^*(t) | < | L(t) - \varLambda(t)\, \Omega_\nu^*(t) | + c_{107} A'^{\frac{1}{g}},$$

par conséquent en vertu de (51), appliqué avec $\Omega_\nu^*(t)$ au lieu de $\Omega_\nu(t)$,

$$\sum_{t=2}^{[N]} | F(t) - \Phi(t)\, \Omega_\nu^*(t) |^2 < c_{108} \{ A'^{-2+\frac{2}{g}} N^3 n^{-gM} + A'^{\frac{2}{g}} N \}$$

$$= c_{109} N^{1+\frac{2}{g}} n^{-M},$$

puisque A' est égal à $N n^{-\frac{1}{2} g M}$.

Ainsi la proposition 9 est démontrée.

Démonstration de la proposition 8.

Il suffit de déduire l'inégalité

$$A < c_{110} N n^{-m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

pour le nombre A des exceptions $\leqq N$ qui sont supérieures à $2K+1$ et supérieures à Nn^{-m}; dans cette démonstration $c_{110}, c_{111}, \ldots, c_{114}$ désignent des nombres positifs, dépendant uniquement de m, ν, K, U et du choix du polynôme $\psi(x)$.

Comme nous l'avons observé dans la communication précédente, $W(p, t)$ est positif pour tout nombre premier p et pour tout entier t appartenant à l'ensemble E. Le lemme 13 nous apprend donc en vertu de (53)

$$\Omega_\nu^*(t) = \prod_{p \leqq (\log t)^\nu} W(p, t) \geqq c_{111} (\nu \log \log t)^{-g}.$$

En outre on a pour tout entier $t \geqq 2K+2$

$$\Phi(t) = K^{-1} g^{-1} b^{-\frac{1}{g}} \varphi^{-1}(U) \sum_{h=2}^{t-2K} \frac{h^{-1+\frac{1}{g}}}{\log(t-h) - \log K} > \frac{c_{112}\, t^{\frac{1}{g}}}{\log t}.$$

Chaque exception t qui est supérieure à $2K+1$ et supérieure à Nn^{-m}, satisfait donc aux inégalités

$$|F(t) - \Phi(t)\, \Omega_\nu^*(t)| \geqq \frac{\Phi(t)\, \Omega_\nu^*(t)}{(\log t)^m} > \frac{c_{113}\, t^{\frac{1}{g}} (\nu \log \log t)^{-g}}{(\log t)^{1+m}}$$

$$> c_{114} N^{\frac{1}{g}} n^{-\frac{m}{g}-1-m} (\log n)^{-g}.$$

Puisque ν est supérieur à $3mg + 2m + 2g$, on peut choisir un nombre M entre $\frac{2m}{g} + 2 + 3m$ et $\frac{\nu}{g}$, de sorte que la proposition 9 nous fournit l'inégalité

$$A\, c_{114}^2 N^{\frac{2}{g}} n^{-\frac{2m}{g}-2-2m} (\log n)^{-2g} < c_{70} N^{1+\frac{2}{g}} n^{-M},$$

qui entraîne (57). Ainsi la proposition 8 est démontrée.

Mathematics. — *Ueber die Beziehungen zwischen den geometrischen Grössen in einer X_n und in einer in der X_n eingebetteten X_m.* Von J. A. SCHOUTEN.

(Communicated at the meeting of May 28, 1938.)

1. *Die verschiedenen Fälle von Einbettung.*

Es seien $x^\varkappa$; $\varkappa, \ldots, \tau = 1, \ldots, n$; Koordinaten in einer X_n und y^a; $a, \ldots, h = 1, \ldots, m$ Koordinaten in einer X_m. Wir können nun bezüglich der Einbettung der X_m in der X_n acht verschiedene Fälle unterscheiden:

Fall 1. Einfache Einbettung ohne weitere Annahmen, d. h. jedem Punkte y^a der X_m ist eindeutig ein Punkt $x^\varkappa$ der X_n zugeordnet vermöge Gleichungen von der Form

$$x^\varkappa = x^\varkappa (y^1, \ldots, y^m). \qquad (1)$$

In jedem Punkte der X_n bestehen zwei Grössen, der Affinor

$$B_b^\varkappa = \partial_b \xi^\varkappa \quad ; \quad \partial_b = \frac{\partial}{\partial \eta^b} \qquad (2)$$

und die einfache kovariante m'-Vektordichte $(m' = n - m)$

$$\mathfrak{t}_{\lambda_1 \ldots \lambda_{m'}} = \frac{1}{m!} \overset{n}{\mathfrak{e}}_{\varkappa_1 \ldots \varkappa_m \lambda_1 \ldots \lambda_{m'}} B_{b_1 \ldots b_m}^{\varkappa_1 \ldots \varkappa_m} \overset{m}{\mathfrak{E}}{}^{b_1 \ldots b_m} \quad ^{1)} \qquad (3)$$

vom Gewicht -1 in X_n und vom Gewicht $+1$ in X_m. Sowohl $B_b^\varkappa$ als $\mathfrak{t}_{\lambda_1 \ldots \lambda_{m'}}$ werden geometrisch dargestellt durch die lokale tangierende E_m.

Fall 1′. Einbettung mit äusserer Orientierung. In jedem Punkte der X_m bekommt die lokale E_m eine „äussere Orientierung", d. h. irgendeine (und folglich jede) E_{n-m} in der lokalen E_n, die keine Richtung mit der lokalen E_m gemeinsam hat, bekommt eine „innere Orientierung", d. h. eine Orientierung schlechthin ($(n-m)$-dimensionaler Schraubsinn). [2]

Eine solche äussere Orientierung wird dadurch festgelegt, dasz man irgend eine Grösse ω vom Absolutwert $+1$ und folgender Transformationsweise

$$\overset{(\varkappa', a')}{\omega} = \frac{\overset{n}{\Delta}}{|\overset{n}{\Delta}|} \frac{\overset{m}{\Delta}}{|\overset{m}{\Delta}|} \overset{(\varkappa, a)}{\omega} \quad ; \quad \overset{n}{\Delta} = Det \left(\frac{\partial x^{\varkappa'}}{\partial x^\varkappa} \right); \overset{m}{\Delta} = Det \left(\frac{\partial y^{a'}}{\partial y^a} \right). \qquad (4)$$

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[1]) Siehe für die Bedeutung von $\overset{n}{\mathfrak{e}}$ und $\overset{m}{\mathfrak{E}}$ SCHOUTEN–STRUIK, Einführung in die neueren Methoden der Differentialgeometrie, I, NOORDHOFF 1935, S. 20.

[2]) „Exterior orientation" und „interior orientation" bei O. VEBLEN und J. H. C. WHITEHEAD, The foundations of differential geometry, Cambridge Tracts, N⁰. 29 (1932), S. 55 und 56.

gibt. Denn $\omega\, \mathfrak{t}_{\lambda_1 \ldots \lambda_{m'}}$ transformiert sich folgendermassen

$$\overset{(\varkappa', a')}{\omega}\, \mathfrak{t}_{\lambda'_1 \ldots \lambda'_{m'}} = |\overset{n}{\triangle}|\ |\overset{m}{\triangle}|^{-1}\, A^{\lambda_1 \ldots \lambda_m}_{\lambda'_1 \ldots \lambda'_m}\ \overset{(\varkappa, a)}{\omega}\, \mathfrak{t}_{\lambda_1 \ldots \lambda_m}, \quad \ldots \quad (5)$$

d. h. genau so wie ein kovarianter m-Vektor der X_n bis auf einen stets positiven Faktor $|\overset{n}{\triangle}|\ |\overset{m}{\triangle}|^{-1}$. Diese Grösse hat also dieselbe Orientierung wie ein solcher kovarianter m-Vektor, und dies ist gerade die gewünschte äussere Orientierung der lokalen E_m [1]).

Fall 2. Einbettung mit Einspannung. In jedem Punkte der X_m ist eine m'-Richtung gegeben, die keine Richtung mit der lokalen E_m gemeinsam hat. Die Einspannung wird gegeben durch einen Affinor $B^{\varkappa}_{\lambda}$, dessen $\varkappa$-Gebiet aus allen kontravarianten Vektoren der lokalen E_m besteht, dessen λ-Gebiet besteht aus allen kovarianten Vektoren, deren $(n-1)$-Richtung die m'-Richtung der Einspannung enthalten, und der der Gleichungen

$$B^{\varkappa}_{\varrho}\, B^{\varrho}_{\lambda} = B^{\varkappa}_{\lambda} \quad \ldots\ldots\ldots\ldots \quad (6)$$

genügt. Aus $B^{\varkappa}_{b}$ und $B^{\varkappa}_{\lambda}$ lässt sich ein Affinor B^{a}_{λ} ableiten vermöge der Gleichungen

$$\left.\begin{aligned} &(\alpha)\quad B^{a}_{\lambda}\, B^{\lambda}_{b} = B^{a}_{b} = \text{Einheitsaffinor der } X_m. \\ &(\beta)\quad B^{b}_{\lambda}\, B^{\varkappa}_{b} = B^{\varkappa}_{\lambda} \end{aligned}\right\} \quad \ldots \quad (7)$$

Das λ-Gebiet von B^{a}_{λ} fällt zusammen mit dem λ-Gebiet von $B^{\varkappa}_{\lambda}$. Im Falle 1 lässt sich B^{a}_{λ} noch nicht eindeutig festlegen, die Gleichung $(7\,\alpha)$ allein lässt nämlich unendlich viele Lösungen zu. Ist aber $v^{\varkappa_1 \ldots \varkappa_p}$ ein einfacher p-Vektor, $p \leqq m$, der in der lokalen E_m liegt, so ist dennoch der Ausdruck $B^{a_1 \ldots a_p}_{\varkappa_1 \ldots \varkappa_p}\, v^{\varkappa_1 \ldots \varkappa_p}$ auch im Falle 1 eindeutig bestimmt.

Fall 2'. Einbettung mit orientierter Einspannung. Dieser Fall entsteht durch Kombination von 1' und 2 und fordert also die Angabe von ω und $B^{\varkappa}_{\lambda}$.

Fall 3. Normalisierte Einbettung mit äusserer Orientierung. In jedem Punkte ist ein kovarianter m'-Vektor $t_{\lambda_1 \ldots \lambda_{m'}}$ gegeben, der die m-Richtung von E_m hat. Es genügt die Grösse

$$\mathfrak{z} = \frac{\mathfrak{t}_{\lambda_1 \ldots \lambda_{m'}}}{t_{\lambda_1 \ldots \lambda_m}} \quad \ldots\ldots\ldots\ldots \quad (8)$$

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1) Nimmt man die Koordinatensysteme so, dass die Orientierung von $\underset{1}{e}^{a}, \ldots, \underset{m}{e}^{a}$ in dieser Reihenfolge gefolgt durch die äussere Orientierung die Orientierung von $\underset{1}{e}^{\varkappa}, \ldots, \underset{n}{e}^{\varkappa}$ in dieser Reihenfolge ergibt, so hat ω i.b. auf diese Koordinationsysteme den Wert $+1$.

zu geben, die sich offenbar folgendermassen transformiert

$$\overset{(\varkappa',a')}{\mathfrak{z}} = \overset{n}{\triangle}\,\overset{m}{\triangle}{}^{-1}\,\overset{(\varkappa,a)}{\mathfrak{z}} \quad \ldots\ldots\ldots \quad (9)$$

und also eine Dichte vom Gewicht -1 in X_n und vom Gewicht $+1$ in X_m ist. Aus (9) folgt

$$\omega = \frac{\mathfrak{z}}{|\mathfrak{z}|} \quad \ldots\ldots\ldots \quad (10)$$

Bei dieser Art der Einbettung ist für jede $E_{m'}$ in der lokalen E_n, die keine Richtung mit der lokalen E_m gemeinsam hat, ein einfacher kontravarianter m'-Vektor $v^{\varkappa_1 \ldots \varkappa_{m'}}$ festgelegt durch die Forderung dass $v^{\varkappa_1 \ldots \varkappa_{m'}}$ in der E_m liegt und der Gleichung

$$v^{\varkappa_1 \ldots \varkappa_{m'}}\, t_{\varkappa_1 \ldots \varkappa_{m'}} = m'! \quad \ldots\ldots\ldots \quad (11)$$

genügt.

Fall 3′. Normalisierte Einbettung ohne Orientierung. Es genügt die Grösse $|\mathfrak{z}|$ zu geben. Dann ist auch das Produkt $\omega\, t_{\lambda_1 \ldots \lambda_{m'}} = |\mathfrak{z}|^{-1}\, \mathfrak{t}_{\lambda_1 \ldots \lambda_{m'}}$ festgelegt aber weder ω noch $t_{\lambda_1 \ldots \lambda_{m'}}$ für sich.

Fall 4. *Normalisierte Einbettung mit orientierter Einspannung.* Dieser Fall entsteht durch Kombination von 3 und 3′ und fordert also die Angabe von $B_\lambda^\varkappa$ und $\mathfrak{z}$. In der m'-Richtung der Einspannung liegt ein m'-Vektor $n^{\varkappa_1 \ldots \varkappa_{m'}}$, der durch die Gleichungen

$$\left.\begin{aligned} &\alpha) \quad n^{\varkappa_1 \ldots \varkappa_{m'}}\, t_{\varkappa_1 \ldots \varkappa_{m'}} = m'! \\ &\beta) \quad B_{\varkappa_1}^{\varkappa}\, n^{\varkappa_1 \ldots \varkappa_{m'}} = 0 \end{aligned}\right\} \quad \ldots\ldots \quad (12)$$

vollständig festgelegt ist und ausserdem der Gleichung

$$t_{\lambda_1 \ldots \lambda_{m'}}\, n^{\varkappa_1 \ldots \varkappa_{m'}} = m'!\, C_{[\lambda_1 \ldots \lambda_{m'}]}^{[\varkappa_1 \ldots \varkappa_{m'}]}\,;\; C_\lambda^\varkappa = A_\lambda^\varkappa - B_\lambda^\varkappa \quad \ldots \quad (13)$$

genügt.

Fall 4′. Normalisierte Einbettung mit Einspannung ohne Orientierung. Dieser Fall entsteht durch Kombination von 2 und 3′ und fordert also die Angabe von $B_\lambda^\varkappa$ und $|\mathfrak{z}|$. Die Produkte $\omega\, t_{\lambda_1 \ldots \lambda_{m'}}$ als $\omega\, n^{\varkappa_1 \ldots \varkappa_{m'}}$ sind festgelegt.

Folgende Tabelle gibt eine Uebersicht der verschiedenen Fälle und der bei ihnen aftretenden Grössen [1]):

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[1]) $+$ steht für „wohl", $-$ für „nicht", t steht für $t_{\lambda_1 \ldots \lambda_{m'}}$, $\mathfrak{t}$ für $\mathfrak{t}_{\lambda_1 \ldots \lambda_{m'}}$ und n für $n^{\varkappa_1 \ldots \varkappa_{m'}}$. Grössen, die sich aus den anderen in der selben Reihe bzw. Spalte vorkommenden ableiten lassen, sind eingeklammert.

Einbettung	1	1′	2	2′	3′	3	4′	4	vorkommende Grössen ausser $B_b^\varkappa$ und t
schliesst ein		1	1	1, 1′, 2	1	1, 1′, 3′	1, 2, 3′	1, 1′, 2, 2′, 3, 3′, 4′	
orientiert	—	+	—	+	—	+	—	+	ω
eingespannt	—	—	+	+	—	—	+	+	$B_\lambda^\varkappa, (B_\lambda^a)$
normalisiert	—	—	—	—	+	+	+	+	$(\omega t), \lvert\mathfrak{z}\rvert$
vorkommende Grössen ausser $B_b^\varkappa$ und t		ω	$B_\lambda^\varkappa, (B_\lambda^a)$	ω $B_\lambda^\varkappa, (B_\lambda^a)$	$(\omega t), \lvert\mathfrak{z}\rvert$	(ω) $(t)\,\mathfrak{z}$	$B_\lambda^\varkappa, (B_\lambda^a)$ $(\omega t), \lvert\mathfrak{z}\rvert$ (ωn)	(ω) $B_\lambda^\varkappa, (B_\lambda^a)$ $(t)\,\mathfrak{z}$ (n)	

2. *Beziehungen zwischen den Grössen der X_n und der X_m.*

A. *Kontravariante p-Vektoren.*

Ein einfacher kontravarianter p-Vektor $v^{\varkappa_1 \dots \varkappa_p}$ der X_n in einem Punkte der X_m bestimmt im Falle 2 für $p \leqq m$ (und für $p = 0$ sogar im Falle 1) eindeutig einen kontravarianten p-Vektor der X_m

$$'v^{a_1 \dots a_p} = B^{a_1 \dots a_p}_{\varkappa_1 \dots \varkappa_p} v^{\varkappa_1 \dots \varkappa_p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Dieser p-Vektor ist die Projektion von $v^{\varkappa_1 \dots \varkappa_p}$ auf die lokale E_m mit der m'-Richtung der Einspannung als Projektionsrichtung. Die Gleichung (14) lässt sich für $p > 0$ nicht nach $v^{\varkappa_1 \dots \varkappa_p}$ lösen. Fängt man aber an mit $'v^{a_1 \dots a_p}$ und definiert man

$$'v^{\varkappa_1 \dots \varkappa_p} = B^{\varkappa_1 \dots \varkappa_p}_{a_1 \dots a_p}\, 'v^{a_1 \dots a_p}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

was sogar im Falle 1 stets möglich ist, so lässt sich diese Gleichung nach $'v^{a_1 \dots a_p}$ lösen:

$$'v^{a_1 \dots a_p} = B^{a_1 \dots a_p}_{\varkappa_1 \dots \varkappa_p}\, 'v^{\varkappa_1 \dots \varkappa_p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

und zwar nicht nur im Falle 2 sondern auch im Falle 1, da die $'v^{\varkappa_1 \dots \varkappa_p}$ in der lokalen E_m liegt.

Für $p \geqq m'$ bestimmt $v^{\varkappa_1 \dots \varkappa_p}$ im Falle 3 noch eine andere Grösse der X_m

$$''v^{a_1 \dots a_{p-m'}} = \frac{1}{m'!} B^{a_1 \dots a_{p-m'}}_{\varkappa_1 \dots \varkappa_{p-m'}} v^{\varkappa_1 \dots \varkappa_p}\, t_{\varkappa_{p-m'+1} \dots \varkappa_p}. \quad . \quad . \quad . \quad (17)$$

Die $(p - m')$-Richtung von $''v^{a_1 \dots a_{p-m'}}$ ist der Schnitt der p-Richtung von $v^{\varkappa_1 \dots \varkappa_p}$ mit der m-Richtung der lokalen E_m. Diese Gleichung kann

nicht nach $v^{\varkappa_1\cdots\varkappa_p}$ gelöst werden. Fängt man aber an mit $''v^{a_1\cdots a_{p-m'}}$ und definiert man

$$''v^{\varkappa_1\cdots\varkappa_p} = \binom{p}{m'} ''v^{a_1\cdots a_{p-m'}} B^{[\varkappa_1\cdots\varkappa_{p-m'}}_{a_1\cdots a_{p-m'}} n^{\varkappa_{p-m'+1}\cdots\varkappa_p]}, \quad . \quad . \quad . \tag{18}$$

was im Falle 4 stets möglich ist (für $p=n$ sogar im Falle 3, da die Unbestimmtheit von $n^{\varkappa_1\cdots\varkappa_{m'}}$ für $p=n$ das rechte Glied nicht beeinflusst), so lässt sich diese Gleichung nach $''v^{a_1\cdots a_{p-m'}}$ lösen:

$$''v^{a_1\cdots a_{p-m'}} = \frac{1}{m'!} B^{a_1\cdots a_{p-m'}}_{\varkappa_1\cdots\varkappa_{p-m'}} ''v^{\varkappa_1\cdots\varkappa_p} t_{\varkappa_{p-m'+1}\cdots\varkappa_p} \cdot \quad . \quad . \quad . \tag{19}$$

Nimmt man im Falle 4 das Koordinatensystem so, dass auf der X_m $\xi^1=\eta^1, \ldots \xi^m=\eta^m$, ist, so ist $\mathfrak{t}_{m+1\ldots n}=1$, $t_{m+1\ldots n}=\mathfrak{z}^{-1}$, und alle Bestimmungszahlen von $\mathfrak{t}_{\lambda_1\ldots\lambda_{m'}}$ und $t_{\lambda_1\ldots\lambda_{m'}}$ mit einem Index von *1* bis m verschwinden [1]). Man kann nun die Richtung der Parameterlinien von $\xi^{m+1}, \ldots, \xi^n$ auf der X_m noch in der m'-Richtung der Einspannung wählen, sodass auch alle Bestimmungszahlen von $n^{\varkappa_1\cdots\varkappa_{m'}}$ mit einem Index von *1* bis m verschwinden und erhält dann

$$n^{m+1\ldots n} = \mathfrak{z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{20}$$

und infolgedessen, wenn wir $\xi^1, \ldots, \xi^m$ als Koordinaten in der X_m benutzen,

$$\left.\begin{aligned} 'v^{\alpha_1\cdots\alpha_p} &= v^{\alpha_1\cdots\alpha_p} \\ ''v^{\alpha_1\cdots\alpha_{p-m'}} &= \mathfrak{z}^{-1} v^{\alpha_1\cdots\alpha_{p-m'}, m+1,\ldots,n}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \tag{21}$$

Man kann das Koordinatensystem sogar noch weiter spezialisieren sodass $\mathfrak{z}=1$ wird. Es ist bemerkenswert, dass für $m \geqq p \geqq m'$ im Falle 4 beide Grössen $'v^{a_1\cdots a_p}$ und $''v^{a_1\cdots a_{p-m'}}$ existieren, und dass diese beiden Grössen, wie aus (21) hervorgeht, für $m=n-1$ (also $m \geqq p \geqq 1$) in diesem Falle $v^{\varkappa_1\cdots\varkappa_p}$ vollständig bestimmen.

Bekanntlich lässt sich $v^{\varkappa_1\cdots\varkappa_p}$ mit Hilfe von $\overset{n}{\mathfrak{e}}_{\lambda_1\ldots\lambda_n}$ als kovariante p'-Vektordichte in X_n darstellen ($p'=n-p$):

$$\mathfrak{v}_{\lambda_1\ldots\lambda_{p'}} = \frac{1}{p!} \overset{n}{\mathfrak{e}}_{\lambda_1\ldots\lambda_{p'}\varkappa_1\ldots\varkappa_p} v^{\varkappa_1\cdots\varkappa_p} \quad . \quad . \quad . \quad . \quad . \quad . \tag{22}$$

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[1]) Die Koordinatensysteme dieser Art gehen auseinander hervor durch die Transformationen einer Gruppe, die von A. KAWAGUCHI betrachtet worden ist. (The foundations of the theory of displacements II, Proc. Imp. Academy, **10**, 45—48 (1934).) Aus dieser Bemerkung ergeben sich die Beziehungen zwischen unseren Betrachtungen und den Untersuchungen von KAWAGUCHI, S. HOKARI (Ueber die Uebertragungen, die der erweiterten Transformationsgruppe angehören, Journ. Hokkaido Imp. Univ., **3**, 15—26 (1935); **4**, 14—50 (1935)) und S. GOLAB (Ueber eine Art der Geometrie von KAWAGUCHI–HOKARI, Ann. Soc. Polon. Math., **16**, 25—30 (1937)) und den in diesen Arbeiten zitierten Untersuchungen von T. HOSOKAWA, A. WUNDHEILER, V. HLAVATY, E. CARTAN, T. Y. THOMAS, J. A. SCHOUTEN und ST. GOLAB.

und ebenso $'v^{a_1\ldots a_p}$ und $''v^{a_1\ldots a_{p-m'}}$ als kovariante $(m-p)$ bzw. $(n-p)$-Vektordichte in X_m

$$'\mathfrak{v}_{b_1\ldots b_{m-p}}=\frac{1}{p!}\overset{m}{\mathfrak{e}}_{b_1\ldots b_{m-p}\,a_1\ldots a_p}\,'v^{a_1\ldots a_p} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$''\mathfrak{v}_{b_1\ldots b_{p'}}=\frac{1}{(m-p')!}\overset{m}{\mathfrak{e}}_{b_1\ldots b_{p'}\,a_1\ldots a_{m-p'}}\,''v^{a_1\ldots a_{m-p'}}. \quad . \quad . \quad . \quad (24)$$

Mit Hilfe dieser Gleichungen lassen sich die Beziehungen zwischen $\mathfrak{v}_{\lambda_1\ldots\lambda_{p'}}$, $'\mathfrak{v}_{\lambda_1\ldots\lambda_{p'}}$, $''\mathfrak{v}_{\lambda_1\ldots\lambda_{p'}}$, $'\mathfrak{v}_{b_1\ldots b_{m-p}}$ und $''\mathfrak{v}_{b_1\ldots b_{p'}}$ aus (14), (15), (16) und (17), (18), (19) ableiten, z.B. $(p\geqq m')$:

$$''\mathfrak{v}_{b_1\ldots b_{p'}}=\mathfrak{z}^{-1}\,B_{b_1\ldots b_{p'}}^{\lambda_1\ldots\lambda_{p'}}\,\mathfrak{v}_{\lambda_1\ldots\lambda_{p'}} \quad \text{(Fall 3)}. \quad . \quad . \quad . \quad . \quad (25)$$

$$\left.\begin{aligned} ''\mathfrak{v}_{\lambda_1\ldots\lambda_{p'}}&=\mathfrak{z}\;B_{\lambda_1\ldots\lambda_{p'}}^{b_1\ldots b_{p'}}\,''\mathfrak{v}_{b_1\ldots b_{p'}} \quad . \quad (26)\\ ''\mathfrak{v}_{b_1\ldots b_{p'}}&=\mathfrak{z}^{-1}\,B_{b_1\ldots b_{p'}}^{\lambda_1\ldots\lambda_{p'}}\,''\mathfrak{v}_{\lambda_1\ldots\lambda_{p'}}. \quad . \quad (27)\end{aligned}\right\}\begin{matrix}\text{(Fall 4, Fall 3}\\ \text{für } p=n)\end{matrix}$$

und (21) geht über in

$$\left.\begin{aligned} '\mathfrak{v}_{\beta_1\ldots\beta_{p'-m'}}&=\mathfrak{v}_{\beta_1\ldots\beta_{p'-m'},\,m+1,\ldots,n}\\ ''\mathfrak{v}_{\beta_1\ldots\beta_{p'}}&=\mathfrak{z}^{-1}\,\mathfrak{v}_{\beta_1\ldots\beta_{p'}}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

B. *Kovariante p-Vektoren.*

Ein einfacher kovarianter p-Vektor $w_{\lambda_1\ldots\lambda_p}$ der X_n in einem Punkte der X_m bestimmt im Falle 1 für $p\leqq m$ einen kovarianten p-Vektor der X_m:

$$'w_{b_1\ldots b_p}=B_{b_1\ldots b_p}^{\lambda_1\ldots\lambda_p}\,w_{\lambda_1\ldots\lambda_p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Dieser p-Vektor ist der Schnitt von $w_{\lambda_1\ldots\lambda_p}$ mit der lokalen E_m. Die Gleichung (29) lässt sich nicht nach $w_{\lambda_1\ldots\lambda_p}$ lösen. Fängt man aber mit $'w_{b_1\ldots b_p}$ an und definiert man

$$'w_{\lambda_1\ldots\lambda_p}=B_{\lambda_1\ldots\lambda_p}^{b_1\ldots b_p}\,'w_{b_1\ldots b_p}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

was im Falle 2 stets möglich ist (für $p=0$ sogar im Falle, 1 da die Unbestimmtheit von B_λ^b für $p=0$ das rechte Glied nicht beeinflusst), so lässt sich die Gleichung (30) nach $'w_{b_1\ldots b_p}$ lösen

$$'w_{b_1\ldots b_p}=B_{b_1\ldots b_p}^{\lambda_1\ldots\lambda_p}\,'w_{\lambda_1\ldots\lambda_p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Die p'-Richtung von $'w_{\lambda_1\ldots\lambda_p}$ enthält die m'-Richtung der Einspannung.

Für $p \geqq m'$ bestimmt $w_{\lambda_1 \ldots \lambda_p}$ im Falle 4 (für $p = n$ sogar im Falle 3) noch eine andere Grösse der X_m

$$''w_{b_1 \ldots b_{p-m'}} = \frac{1}{m!} B^{\lambda_1 \ldots \lambda_{p-m'}}_{b_1 \ldots b_{p-m'}} w_{\lambda_1 \ldots \lambda_p} n^{\lambda_{p-m'+1} \ldots \lambda_p} \quad . \quad . \quad . \quad (32)$$

Die $(n-p)$-Richtung von $''w_{b_1 \ldots b_{p-m'}}$ ist die Projektion der $(n-p)$-Richtung von $w_{\lambda_1 \ldots \lambda_p}$ auf die lokale E_m mit der m'-Richtung der Einspannung als Projektionsrichtung. Diese Gleichung kann für $p \neq n$ nicht nach $w_{\lambda_1 \ldots \lambda_p}$ gelöst werden. Fängt man aber mit $''w_{b_1 \ldots b_{p-m'}}$ an und definiert man

$$''w_{\lambda_1 \ldots \lambda_p} = \binom{p}{m'} ''w_{b_1 \ldots b_{p-m'}} B^{b_1 \ldots b_{p-m'}}_{[\lambda_1 \ldots \lambda_{p-m'}} t_{\lambda_{p-m'+1} \ldots \lambda_p]}, \quad . \quad . \quad (33)$$

was im Falle 3 stets möglich ist, so lässt sich diese Gleichung nach $''w_{b_1 \ldots b_{p-m'}}$ lösen

$$''w_{b_1 \ldots b_{p-m'}} = \frac{1}{m!} B^{\lambda_1 \ldots \lambda_{p-m'}}_{b_1 \ldots b_{p-m'}} ''w_{\lambda_1 \ldots \lambda_p} n^{\lambda_{p-m'+1} \ldots \lambda_p} \quad . \quad . \quad . \quad (34)$$

und zwar nicht nur im Falle 4 sondern auch im Falle 3, da die Unbestimmtheit von $n^{\varkappa_1 \ldots \varkappa_{m'}}$ das rechte Glied nicht beeinflusst.

Nimmt man im Falle 4 das besondere oben eingeführte Koordinatensystem, so gehen (29) und (32) über in

$$\left.\begin{aligned} 'w_{\beta_1 \ldots \beta_p} &= w_{\beta_1 \ldots \beta_p} \\ ''w_{\beta_1 \ldots \beta_{p-m'}} &= \mathfrak{z}\, w_{\beta_1 \ldots \beta_{p-1}, m+1, \ldots, n} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Auch hier existieren für $m \geqq p \geqq m'$ im Falle 4 beide Grössen $'w_{b_1 \ldots b_p}$ und $''w_{b_1 \ldots b_{p-m'}}$ und aus (35) folgt, dass diese beiden Grössen zusammen für $m = n-1$ (also $m \geqq p \geqq 1$) in diesem Falle $w_{\lambda_1 \ldots \lambda_p}$ vollständig bestimmen.

Vollziehen wir wieder den Uebergang zu den kontravarianten Dichten in X_n und X_m so lassen sich die mit (29), (30), (31) und (32), (33), (34) korrespondierenden Formeln ableiten, z. B. $(p \geqq m')$:

$$''\mathfrak{w}^{a_1 \ldots a_{p'}} = \mathfrak{z}\, B^{a_1 \ldots a_{p'}}_{\varkappa_1 \ldots \varkappa_{p'}} \mathfrak{w}^{\varkappa_1 \ldots \varkappa_{p'}} \quad \text{(Fall 4, Fall 3 für } p = n) \quad (36)$$

$$\left.\begin{aligned} ''\mathfrak{w}^{\varkappa_1 \ldots \varkappa_{p'}} &= \mathfrak{z}^{-1} B^{\varkappa_1 \ldots \varkappa_{p'}}_{a_1 \ldots a_{p'}} ''\mathfrak{w}^{a_1 \ldots a_{p'}} \quad . \quad . \quad . \quad . \quad (37) \\ ''\mathfrak{w}^{a_1 \ldots a_{p'}} &= \mathfrak{z}\, B^{a_1 \ldots a_{p'}}_{\varkappa_1 \ldots \varkappa_{p'}} ''\mathfrak{w}^{\varkappa_1 \ldots \varkappa_{p'}} . \quad . \quad . \quad . \quad . \quad (38) \end{aligned}\right\} \text{(Fall 3)}$$

und (35) geht über in

$$\left.\begin{aligned} '\mathfrak{w}^{\alpha_1 \ldots \alpha_{p'-m'}} &= \mathfrak{w}^{\alpha_1 \ldots \alpha_{p'-m'}, m+1, \ldots, n} \\ ''\mathfrak{w}^{\alpha_1 \ldots \alpha_{p'}} &= \mathfrak{z}\, \mathfrak{w}^{\alpha_1 \ldots \alpha_{p'}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Folgende Tabelle gibt eine Uebersicht [1])

X_n	Fall		X_m	X_n	Fall		X_m
$p \leqq m$	1	2		$p \geqq m'$	3	4	
$v^{\varkappa_1\ldots\varkappa_p}$	$p=0$	B^a_λ	$'v^{a_1\ldots a_p}$	$v^{\varkappa_1\ldots\varkappa_p}$	$(B^a_\lambda), t$	B^a_λ, t	$''v^{a_1\ldots a_{p-m'}}$
$\mathfrak{v}_{\lambda_1\ldots\lambda_{p'}}$	$(\rightleftarrows)$	$\rightarrow$	$'\mathfrak{v}_{b_1\ldots b_{m-p}}$	$\mathfrak{v}_{\lambda_1\ldots\lambda_{p'}}$	$\rightarrow$	$\rightarrow$	$''\mathfrak{v}_{b_1\ldots b_{p'}}$
$'v^{\prime_1\ldots\prime_p}$	$B^\varkappa_b$ $\leftrightarrows$	$B^\varkappa_b$ $\leftrightarrows$	$'v^{a_1\ldots a_p}$	$''v^{\prime_1\ldots\prime_p}$	$p=n$ $(\leftrightarrows)$	$B^\varkappa_b, n$ $\leftrightarrows$	$''v^{a_1\ldots a_{p-m'}}$
$'\mathfrak{v}_{\lambda_1\ldots\lambda_{p'}}$	(B^a_λ)	B^a_λ	$'\mathfrak{v}_{b_1\ldots b_{m\ p}}$	$''\mathfrak{v}_{\lambda_1\ldots\lambda_{p'}}$		B^a_λ, t	$''\mathfrak{v}_{b_1\ldots b_{p'}}$
$w_{\lambda_1\ldots\lambda_p}$	$B^\varkappa_b$	$B^\varkappa_b$	$'w_{b_1\ldots b_p}$	$w_{\lambda_1\ldots\lambda_p}$	$p=n$	$B^\varkappa_b, n$	$''w_{b_1\ldots b_{p-m'}}$
$\mathfrak{w}^{\varkappa_1\ldots\varkappa_{p'}}$	$\rightarrow$	$\rightarrow$	$'\mathfrak{w}^{a_1\ldots a_{m-p}}$	$\mathfrak{w}^{\varkappa_1\ldots\varkappa_{p'}}$	$(\rightleftarrows)$		$''\mathfrak{w}^{a_1\ldots a_{p'}}$
$'w_{\lambda_1\ldots\lambda_p}$	$p=0$ $(\leftrightarrows)$	B^a_λ $\leftrightarrows$	$'w_{b_1\ldots b_p}$	$''w_{\lambda_1\ldots\lambda_p}$	$(B^a_\lambda), t$ $\leftrightarrows$	B^a_λ, t $\leftrightarrows$	$''w_{b_1\ldots b_{p-m'}}$
$'\mathfrak{w}^{\prime_1\ldots\prime_{p'}}$		$B^\varkappa_b$	$'\mathfrak{w}^{a_1\ldots a_{m-p}}$	$''\mathfrak{w}^{\varkappa_1\ldots\varkappa_p}$	$B^\varkappa_b, (n)$	$B^\varkappa_b, n$	$''\mathfrak{w}^{a_1\ldots a_{p'}}$

Die hier vorkommenden Grössen sind, als Dichten geschrieben, gewöhnliche Dichten, d. h. solche, die sich mit einer Potenz von $\triangle$ transformieren. Betrachtet man statt dessen die sogenannten WEYLschen Dichten, 'die sich mit einer Potenz von $|\triangle|$ transformieren und die mit Ihnen vermöge $\overset{n}{\mathfrak{E}}$, $\overset{n}{\mathfrak{e}}$, $\overset{m}{\mathfrak{E}}$ und $\overset{m}{\mathfrak{e}}$ korrespondierenden Grössen, so treten die Fälle $1'$, $2'$, $3'$ $4'$ an Stelle von 1, 2, 3, 4. Sonst ändert sich an der Uebersichtstabelle nichts. Auch alle Formeln bleiben erhalten, nur bekommt die eine Seite jeder Formel einen Faktor ω. Auf die geometrische Bedeutung solcher Grössen und ihre Beziehungen i. b. auf X_m und X_n soll in einer anderen Arbeit näher eingegangen werden.

[1]) Die verwendeten Grössen sind bei den Pfeilen angegeben. Grössen, die im vorliegenden Falle unbestimmt sind, die aber doch verwendet werden können, da das Resultat von der Unbestimmtheit nicht beeinflusst wird, sind eingeklammert, t steht für $t_{\lambda_1\ldots\lambda_{p'}}$ und n für $n^{\varkappa_1\ldots\varkappa_{p'}}$.

Mathematics. — *Zur Differentialgeometrie der Gruppe der Berührungstransformationen.* IV. *Kovariante Ableitungen in der* K_{2n-1}[1]. Von J. A. SCHOUTEN und J. HAANTJES.

(Communicated at the meeting of May 28, 1938.)

Es seien $x^{\varkappa}$, $\varkappa = 0, 1, \ldots, n$ homogene Koordinaten in einem n-dimensionalen Raum und p_{λ}, $\lambda = 0, 1, \ldots, n$ homogene lokale Facettenkoordinaten. $\lfloor x^{\varkappa} \rfloor$,[2] stellt einen *Punkt* dar, die Kombination $\lfloor x^{\varkappa} \rfloor$, $\lfloor p_{\lambda} \rfloor$, sofern $x^{\varrho} p_{\varrho} = 0$ ist, ein *Element*. Die Kombination $x^{\varkappa}$, p_{λ} heisst das zum Element gehörige *analytische Element*. Die Gesamtheit aller Elemente bildet eine $(2n-1)$-dimensionale Mannigfaltigkeit mit $2n+2$ homogenen Koordinaten zwischen denen eine Beziehung besteht.

Zwei benachbarte Elementen $\lfloor x^{\varkappa} \rfloor$, $\lfloor p_{\lambda} \rfloor$ und $\lfloor x^{\varkappa} + dx^{\varkappa} \rfloor$, $\lfloor p_{\lambda} + dp_{\lambda} \rfloor$ liegen *vereinigt*, wenn $p_{\varrho}\, dx^{\varrho} = 0$ also auch $x^{\varrho}\, dp_{\varrho} = 0$ ist. Eine Menge von Elementen von denen je zwei benachbarte vereinigt liegen, heisst *Elementverein*. Jede Transformation von Elementen, die jeden Elementverein in einen Elementverein überführt und eine eindeutige Umkehrung besitzt, heisst *Berührungstransformation*. Analytisch ist eine solche Transformation charakterisiert durch die Invarianz der Gleichung $p_{\varrho}\, x^{\varrho} = 0$ und des Gleichungssystems $p_{\varrho}\, x^{\varrho} = 0$, $p_{\varrho}\, dx^{\varrho} = 0$. Dies sind *Objekttransformationen* (Koordinaten unverändert, Objekte transformiert). Auch die *Koordinatentransformationen* (Objekte unverändert, Koordinaten transformiert), die durch dieselbe Invarianz charakterisiert sind, nennen wir Berührungstransformationen. In (K I)[3] ist bewiesen, dass jede Koordinatentransformation, die eine Berührungstransformation ist, sich in folgender Weise schreiben lässt

$$\mathfrak{K}_{2n+2} \begin{cases} x^{\varkappa'} = \varphi^{\varkappa'}(x^{\varrho}, p_{\sigma}) \\ \\ p_{\lambda'} = \psi_{\lambda'}(x^{\varrho}, p_{\sigma}) \end{cases} ; \quad \begin{vmatrix} \dfrac{\partial \varphi^{\varkappa'}}{\partial x^{\varkappa}} & \dfrac{\partial \varphi^{\varkappa'}}{\partial p_{\lambda}} \\ \\ \dfrac{\partial \psi_{\lambda'}}{\partial x^{\varkappa}} & \dfrac{\partial \psi_{\lambda'}}{\partial p_{\lambda}} \end{vmatrix} \neq 0 \quad . \; . \; . \; . \quad (1)$$

[1]) Anlass zu dieser Untersuchung bildete die Arbeit Invariant theory of homogeneous contracttransformations von L. P. EISENHART und M. S. KNEBELMANN, wo zum erstenmale kovariante Ableitungen in der Elementmannigfaltigkeit betrachtet wurden. Unsere Behandlung unterscheidet sich dadurch, dass wir von doppelthomogenen Kontakttransformationen ausgehen und bis zu dem fundamentalen Theorem vordringen, das gestattet jeden linearen Zusammenhang mit Hilfe von gewissen Kontaktaffinoren festzulegen.

[2]) $\lfloor \; \rfloor$ bedeutet: abgesehen von einem beliebigen Zahlenfaktor.

[3]) J. A. SCHOUTEN, Zur Differentialgeometrie der Gruppe der Berührungstransformationen, I. Doppelthomogene Behandlung von Berührungstransformationen, Proc. Kon. Akad. v. Wetensch., Amsterdam, 40, 100—107 (1937).

wo die $\varphi^{\varkappa'}$ und $\psi_{\lambda'}$ homogene Funktionen ersten bzw. nullten Grades in $x^{\varkappa}$ und nullten bzw. ersten Grades in p_λ sind und folgenden Bedingungen genügen:

$$\left.\begin{array}{c} V_{\varrho'[\mu} T^{\varrho'}_{\lambda]} = 0 \;;\; T^{[\nu}_{\varrho'} U^{\varkappa]\varrho'} = 0 \;;\; V_{\varrho[\mu'} T^{\varrho}_{\lambda']} = 0 \;;\; T^{[\nu'}_{\varrho} U^{\mu']\varrho} = 0 \\ T^{\varkappa}_{\varrho'} T^{\varrho'}_{\lambda} - V_{\lambda\varrho'} U^{\varkappa\varrho'} = A^{\varkappa}_{\lambda} \;;\; T^{\varkappa'}_{\varrho} T^{\varrho}_{\lambda'} - V_{\lambda'\varrho} U^{\varkappa'\varrho} = A^{\varkappa'}_{\lambda'} \end{array}\right\} . \quad (2)$$

$$\left.\begin{array}{c} p_{\lambda'} = T^{\lambda}_{\lambda'} p_\lambda \;;\; p_\lambda = T^{\lambda'}_{\lambda} p_{\lambda'} \\ x^{\varkappa'} = T^{\varkappa'}_{\varkappa} x^{\varkappa} \;;\; x^{\varkappa} = T^{\varkappa}_{\varkappa'} x^{\varkappa'} \\ U^{\varkappa'\lambda} p_\lambda = 0 \;;\; U^{\varkappa\lambda'} p_{\lambda'} = 0 \\ V_{\lambda'\varkappa} x^{\varkappa} = 0 \;;\; V_{\lambda\varkappa'} x^{\varkappa'} = 0 \end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

$$\left.\begin{array}{c} T^{\varkappa}_{\lambda'} = \partial_{\lambda'} x^{\varkappa} = \partial^{\varkappa} p_{\lambda'} \;;\; T^{\varkappa'}_{\lambda} = \partial_\lambda x^{\varkappa'} = \partial^{\varkappa'} p_\lambda \;;\; \partial_\lambda = \dfrac{\partial}{\partial x^\lambda};\; \partial^\lambda = \dfrac{\partial}{\partial p_\lambda} \\ U^{\nu\varkappa'} = - U^{\varkappa'\nu} = \partial^\nu x^{\varkappa'} = - \partial^{\varkappa'} x^\nu \;;\; V_{\mu\lambda'} = - V_{\lambda'\mu} = \partial_\mu p_{\lambda'} = - \partial_{\lambda'} p_\mu . \end{array}\right\} . \quad (4)$$

Diese Transformationen bilden eine Gruppe $\mathfrak{K}_{2n+2}$. Bei dieser Gruppe ist der Ausdruck $p_\varrho x^\varrho$ invariant und unter der Bedingung $p_\varrho x^\varrho = 0$ sowohl der Ausdruck $x^\varrho dp_\varrho$ als auch $p_\varrho dx^\varrho$.

Daneben betrachten wir die Transformationen der analytischen Elemente

$$\mathfrak{F}: \; 'x^{\varkappa} = \varrho x^{\varkappa} \;;\; 'p_\lambda = \varrho^{-1} p_\lambda , \quad . \; . \; . \; . \; . \; . \; . \quad (5)$$

wo ϱ eine homogene Funktion nullten Grades der $x^{\varkappa}$ und p_λ ist. Diese Transformationen lassen jedes einzelne Element invariant und ausserdem sowohl $p_\varrho x^\varrho$ als auch unter der Bedingung $p_\varrho x^\varrho = 0$ noch $x^\varrho dp_\varrho$ und $p_\varrho dx^\varrho$.

Sie bilden eine Gruppe $\mathfrak{F}$. Die Mannigfaltigkeit aller Elemente, ausgestattet mit den Gruppen $\mathfrak{K}_{2n+2}$ und $\mathfrak{F}$ heisse K_{2n-1}.

Die Grössen in K_{2n-1}.

Statt p_λ schreiben wir im folgenden auch $x^{(\lambda)}$ und lassen die Indizes $\mathfrak{a}, \ldots, \mathfrak{g}$ die $2n+2$ Werte $1, \ldots, n+1, (1), \ldots, (n+1)$ durchlaufen. Sodann schreiben wir

$$\frac{\partial x^{\mathfrak{a}'}}{\partial x^{\mathfrak{b}}} = A^{\mathfrak{a}'}_{\mathfrak{b}} , \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

sodass

$$\left.\begin{array}{c} A^{\varkappa'}_{\lambda} = A^{(\lambda)}_{(\varkappa')} = T^{\varkappa'}_{\lambda} \;;\; A^{\varkappa'}_{(\lambda)} = - A^{\lambda}_{(\varkappa')} = U^{\lambda\varkappa'} = - U^{\varkappa'\lambda} \\ A^{(\varkappa')}_{\lambda} = - A^{(\lambda)}_{\varkappa'} = V_{\lambda\varkappa'} = - V_{\varkappa'\lambda} \;;\; A^{(\varkappa')}_{(\lambda)} = A^{\lambda}_{\varkappa'} = T^{\lambda}_{\varkappa'} \end{array}\right\} \quad . \; . \quad (7)$$

ist und definieren nun folgenden Grössen

1. *Skalare* mit der Gradzahl $\mathfrak{r}$, homogen $^1/_2\,\mathfrak{r}$-ten Grades in $x^{\varkappa}$ und in p_λ, invariant bei $\mathfrak{K}_{2n+2}$ und $\mathfrak{F}$.

2. *Kontravariante Kontaktvektoren* mit der Gradzahl $\mathfrak{r}$ mit $2n+2$ Bestimmungszahlen $v^{\varkappa}$, $v_\lambda = v^{(\lambda)}$ homogen von den Graden $^1/_2(\mathfrak{r}+1)$, $^1/_2(\mathfrak{r}-1)$ bzw. $^1/_2(\mathfrak{r}-1)$, $^1/_2(\mathfrak{r}+1)$ in $x^{\varkappa}$, p_λ und den Transformationsgleichungen

$$\left.\begin{array}{l} \mathfrak{K}_{2n+2}: \ v^{\mathfrak{a}'} = A^{\mathfrak{a}'}_{\mathfrak{b}} v^{\mathfrak{b}} \ \text{oder} \left\{\begin{array}{l} v^{\varkappa'} = T^{\varkappa'}_{\varkappa} v^{\varkappa} + U^{\lambda\varkappa'} v_\lambda \\ v_{\lambda'} = V_{\varkappa\lambda'} v^{\varkappa} + T^{\lambda}_{\lambda'} v_\lambda \end{array}\right. \\ \mathfrak{F}: \left\{\begin{array}{l} 'v^{\varkappa'} = \varrho\, v^{\varkappa} \\ 'v_\lambda = \varrho^{-1} v_\lambda \end{array}\right. \end{array}\right\} \quad . \; . \; . \quad (8)$$

3. *Kovariante Kontactxektoren* mit der Gradzahl $\mathfrak{r}$ mit $2n+2$ Bestimmungszahlen w_λ, $-w^{\varkappa} = w_{(\varkappa)}$ von den Graden $^1/_2(\mathfrak{r}-1)$, $^1/_2(\mathfrak{r}+1)$ und $^1/_2(\mathfrak{r}+1)$, $^1/_2(\mathfrak{r}-1)$ in $x^{\varkappa}$, p_λ und den Transformationsgleichungen

$$\left.\begin{array}{l} \mathfrak{K}_{2n+2}: \ w_{\mathfrak{b}'} = A^{\mathfrak{a}}_{\mathfrak{b}'} w_{\mathfrak{a}} \ \text{oder} \left\{\begin{array}{l} w_{\lambda'} = T^{\lambda}_{\lambda'} w_\lambda + V_{\varkappa\lambda'} w^{\varkappa} \\ w^{\varkappa'} = U^{\lambda\varkappa'} w_\lambda + T^{\varkappa'}_{\varkappa} w^{\varkappa} \end{array}\right. \\ \mathfrak{F}: \left\{\begin{array}{l} 'w_\lambda = \varrho^{-1} w_\lambda \\ 'w^{\varkappa} = \varrho\, w^{\varkappa} \end{array}\right. \end{array}\right\} \quad . \; . \quad (9)$$

Korollar[1]): *a.* Ist $v^{\varkappa}, v_\lambda$ ein kontravarianter Vektor mit der Gradzahl $\mathfrak{r}$, so ist $v_\lambda, -v^{\varkappa}$ ein kovarianter Vektor mit der Gradzahl $\mathfrak{r}$.

b. Infolge $V_{\varkappa\lambda'} x^{\varkappa} = 0$; $U^{\lambda\varkappa'} p_\lambda = 0$ sind $x^{\varkappa}, 0$; $0, p_\lambda$; $x^{\varkappa}$, p_λ und $-x^{\varkappa}, p_\lambda$ kontravariante Vektoren und $0, x^{\varkappa}$; $p_\lambda, 0$; $p_\lambda, x^{\varkappa}$ und $-p_\lambda, x^{\varkappa}$ kovariante Vektoren mit der Gradzahl 1.

4. *Kontaktaffinoren und Dichten*, wie üblich als Summe von Produkten von Kontaktvektoren. Die Gradzahl ist das Produkt der Gradzahlen der Faktoren. Eine Bestimmungszahl eines Affinors $P^{\mathfrak{a}_1 \ldots \mathfrak{a}_p}_{\mathfrak{b}_1 \ldots \mathfrak{b}_q}$ mit der Gradzahl $\mathfrak{r}$, die oben p_1 nichteingeklammerte und $p_2 = p - p_1$ eingeklammerte Indizes hat und unten q_1 nichteingeklammerte und $q_2 = q - q_1$ eingeklammerte Indizes, ist homogen vom Grade $^1/_2\mathfrak{r} + {}^1/_2(p_1 - p_2 - q_1 + q_2)$ in $x^{\varkappa}$ und vom Grade $^1/_2\mathfrak{r} - {}^1/_2(p_1 - p_2 - q_1 + q_2)$ in p_λ und bekommt also bei $\mathfrak{F}$ einen Faktor $\varrho^{p_1 - p_2 - q_1 + q_2}$. Die Gradzahl ist also die Summe der Grade in $x^{\varkappa}$ und p_λ von jeder Bestimmungszahl. Daraus folgt, dass

$$x^{\mathfrak{c}}\, \partial_{\mathfrak{c}} P^{\mathfrak{a}_1 \ldots \mathfrak{a}_p}_{\mathfrak{b}_1 \ldots \mathfrak{b}_q} = \mathfrak{r}\, P^{\mathfrak{a}_1 \ldots \mathfrak{a}_p}_{\mathfrak{b}_1 \ldots \mathfrak{b}_q}$$

ist und dass die Gradzahl einer Ueberschiebung die Summe der Gradzahlen der Faktoren ist. Die Gradzahl ist also invariant bei Faltung („Verjüngung"). Die Gradzahl des Einheitsaffinors $A^{\mathfrak{a}}_{\mathfrak{b}}$ ist Null. Da die Gradzahl nichts mit der Transformation bei $\mathfrak{K}_{2n+2}$ zu tun hat, lässt sich

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[1]) Diese Eigenschaften gehen zum Teil verloren, wenn wir (5) die allgemeineren Transformationen $'x^{\varkappa} = \varrho\, x^{\varkappa}$; $'p_\lambda = \sigma\, p_\lambda$ nehmen, wie dies in (K I) geschehen ist.

ihre Definition verallgemeineren für geometrische Objekte mit oberen und unteren Indizes, die keine Affinoren sind. Z. B. ist die Gradzahl von $\partial_{\mathfrak{c}} P^{\mathfrak{a}_1 \ldots \mathfrak{a}_p}_{\mathfrak{b}_1 \ldots \mathfrak{b}_q}$ gleich $\mathfrak{r}-1$.

5. *Der kontravariante Fundamentalbivektor* $f^{\mathfrak{a}\mathfrak{b}}$ mit den Bestimmungszahlen

$$f^{\varkappa\lambda}=0;\quad f^{\varkappa(\lambda)}=-f^{(\lambda)\varkappa}=\delta^{\varkappa}_{\lambda};\quad f^{(\varkappa)(\lambda)}=0 \quad \ldots\ldots \quad (10)$$

i. b. auf irgend ein Bezugssystem und infolgedessen auch i. b. auf jedes andere Bezugssystem. Die Gradzahl ist Null und die Bestimmungszahlen sind invariant bei $\mathfrak{F}$.

6. *Der kovariante Fundamentalbivektor* $f_{\mathfrak{b}\mathfrak{a}}$ mit den Bestimmungszahlen

$$f_{\lambda\varkappa}=0;\quad f_{\lambda(\varkappa)}=-f_{(\varkappa)\lambda}=\delta^{\varkappa}_{\lambda};\quad f_{(\lambda)(\varkappa)}=0 \quad \ldots\ldots \quad (11)$$

i. b. auf jedes Bezugssystem. Die Gradzahl ist Null und auch diese Bestimmungszahlen sind invariant bei $\mathfrak{F}$.

Es gilt die Gleichung

$$f^{\mathfrak{a}\mathfrak{c}} f_{\mathfrak{b}\mathfrak{c}} = A^{\mathfrak{a}}_{\mathfrak{b}} \quad \ldots\ldots\ldots\ldots \quad (12)$$

Wir verwenden $f^{\mathfrak{a}\mathfrak{b}}$ und $f_{\mathfrak{b}\mathfrak{a}}$ zum Herauf- und Herunterziehen von Indizes und verabreden zur Ueberschiebung bei $f^{\mathfrak{a}\mathfrak{b}}$ stets den *ersten* und bei $f_{\mathfrak{b}\mathfrak{a}}$ stets den *zweiten* Index zu benutzen.

Dann ist

$$\left.\begin{array}{ll} v_{\mathfrak{b}} = f_{\mathfrak{b}\mathfrak{a}}\, v^{\mathfrak{a}} \ \text{ oder } & \left\{\begin{array}{l} v_{\lambda} = f_{\lambda(\varkappa)}\, v^{(\varkappa)} = v^{(\lambda)} \\ v_{(\lambda)} = f_{(\lambda)\varkappa}\, v^{\varkappa} = -v^{\varkappa} \end{array}\right. \\ v^{\mathfrak{a}} = v_{\mathfrak{b}}\, f^{\mathfrak{b}\mathfrak{a}} \ \text{ oder } & \left\{\begin{array}{l} v^{\varkappa} = v_{(\lambda)}\, f^{(\lambda)\varkappa} = -v_{(\lambda)} \\ v^{(\lambda)} = v_{\varkappa}\, f^{\varkappa(\lambda)} = v_{\lambda} \end{array}\right. \end{array}\right\} \quad \ldots\ldots \quad (13)$$

in Uebereinstimmung mit der unter 2. und 3. getroffenen Festsetzung.

Aus (12) folgt

$$f^{\mathfrak{a}}_{\cdot\mathfrak{b}} = -f_{\mathfrak{b}}^{\cdot\mathfrak{a}} = A^{\mathfrak{a}}_{\mathfrak{b}} \quad \text{oder} \quad \left.\begin{array}{l} f^{\varkappa}_{\cdot\lambda} = -f_{\lambda}^{\cdot\varkappa} = \delta^{\varkappa}_{\lambda} \\ f^{(\varkappa)}_{\cdot(\lambda)} = -f_{(\lambda)}^{\cdot(\varkappa)} = \delta^{(\varkappa)}_{(\lambda)}. \end{array}\right\} \quad \ldots \quad (14)$$

Allgemein gilt die Regel, dass bei dem Herauf- und Herunterziehen Vorzeichenwechsel dann und nur dann auftritt, wenn der obere Index nicht eingeklammert und der untere also eingeklammert ist.[1]) Wie nach der Einführung eines Fundamentaltensors ist auch hier der Unterschied zwischen ko- und kontravarianten Grössen nach Einführung des Fundamentalbivektors verschwunden und bleibt nur der Unterschied zwischen ko-, kontravarianten und gemischten Bestimmungszahlen. Anders als in der

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1) Es ist zu beachten, dass also stets $v^{\mathfrak{a}} w_{\mathfrak{a}} = -v_{\mathfrak{a}} w^{\mathfrak{a}}$.

RIEMANNschen Geometrie unterscheiden sich die verschiedenen Arten von Bestimmungszahlen höchstens durch das Vorzeichen, da sich bei dem Herauf- und Herunterziehen der Indizes der Absolutwert nicht ändert.

Zusammenhang in der K_{2n-1}.

Ein linearer Zusammenhang in der K_{2n-1} ist festgelegt durch die Gleichung

$$\nabla_{\mathfrak{c}}\, v^{\mathfrak{a}} = \partial_{\mathfrak{c}}\, v^{\mathfrak{a}} + \Pi^{\mathfrak{a}}_{\mathfrak{c}\mathfrak{b}}\, v^{\mathfrak{b}}, \quad \ldots\ldots\ldots \quad (15)$$

wo die $\Pi^{\mathfrak{a}}_{\mathfrak{c}\mathfrak{b}}$ ein geometrisches Objekt mit der Gradzahl —1 bilden mit der Transformationsweise

$$\mathfrak{K}_{2n+2}\colon \quad \Pi^{\mathfrak{a}'}_{\mathfrak{c}'\mathfrak{b}'} = A^{\mathfrak{c}\mathfrak{b}\mathfrak{a}'}_{\mathfrak{c}'\mathfrak{b}'\mathfrak{a}}\, \Pi^{\mathfrak{a}}_{\mathfrak{c}\mathfrak{b}} + A^{\mathfrak{a}'}_{\mathfrak{b}}\, \partial_{\mathfrak{c}'}\, A^{\mathfrak{b}}_{\mathfrak{b}'} \quad \ldots\ldots \quad (16a)$$

$$\mathfrak{F}\colon \quad \left.\begin{array}{llll} & \text{Grade} & & \text{Grade} \\ '\Pi^{\varkappa}_{\mu\lambda} = \varrho^{-1}\, \Pi^{\varkappa}_{\mu\lambda}\,; & (-1, 0); & '\Pi^{(\varkappa)}_{(\mu)(\lambda)} = \varrho\, \Pi^{(\varkappa)}_{(\mu)(\lambda)}\,; & (0, -1) \\ '\Pi^{(\varkappa)}_{\mu\lambda} = \varrho^{-3}\, \Pi^{(\varkappa)}_{\mu\lambda}\,; & (-2, 1); & '\Pi^{\varkappa}_{(\mu)(\lambda)} = \varrho^{3}\, \Pi^{\varkappa}_{(\mu)(\lambda)}\,; & (1, -2) \\ '\Pi^{(\varkappa)}_{(\mu)\lambda} = \varrho^{-1}\, \Pi^{(\varkappa)}_{(\mu)\lambda}\,; & (-1, 0); & '\Pi^{\varkappa}_{\mu(\lambda)} = \varrho\, \Pi^{\varkappa}_{\mu(\lambda)}\,; & (0, -1) \\ '\Pi^{(\varkappa)}_{\mu(\lambda)} = \varrho^{-1}\, \Pi^{(\varkappa)}_{\mu(\lambda)}\,; & (-1, 0); & '\Pi^{\varkappa}_{(\mu)\lambda} = \varrho\, \Pi^{\varkappa}_{(\mu)\lambda}\,; & (0, -1) \end{array}\right\} . \quad (16b)$$

Bei kovarianter Differentiation vermindert die Gradzahl um 1. Da $dx^{\mathfrak{a}}$ kein Kontaktvektor ist, gibt es im allgemeinen kein kovariantes Differential. Wie in einer H_{2n-1} leiten sich aus den $\Pi^{\mathfrak{a}}_{\mathfrak{c}\mathfrak{b}}$ drei Affinoren ab [1])

$$\left.\begin{array}{l} P^{\mathfrak{a}}_{.\mathfrak{b}} \overset{h}{=} x^{\mathfrak{c}}\, \Pi^{\mathfrak{a}}_{\mathfrak{c}\mathfrak{b}} + A^{\mathfrak{a}}_{\mathfrak{b}} \\ Q^{\mathfrak{a}}_{.\mathfrak{c}} \overset{h}{=} \Pi^{\mathfrak{a}}_{\mathfrak{c}\mathfrak{b}}\, x^{\mathfrak{b}} + A^{\mathfrak{a}}_{\mathfrak{c}} \end{array}\right\} \text{Gradzahl } 0 \quad \ldots\ldots\ldots \quad (17)$$

$$S_{\mathfrak{c}\mathfrak{b}}^{\cdot\cdot\,\mathfrak{a}} \overset{h}{=} \Pi_{[\mathfrak{c}\mathfrak{b}]}^{\cdot\cdot\,\mathfrak{a}}\,; \quad \text{Gradzahl } -1 \quad \ldots\ldots\ldots \quad (18)$$

Schreiben wir

$$\nabla_{\mathfrak{c}}\, f_{\mathfrak{b}\mathfrak{a}} = F_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}\,; \quad F_{\mathfrak{c}(\mathfrak{b}\mathfrak{a})} = 0, \quad \ldots\ldots\ldots \quad (19)$$

so folgt

$$2\, \Pi^{\mathfrak{b}}_{\mathfrak{c}[\mathfrak{b}}\, f_{\mathfrak{a}]\mathfrak{b}} = F_{\mathfrak{c}\mathfrak{b}\mathfrak{a}} \quad \ldots\ldots\ldots \quad (20)$$

oder

$$2\, \Pi_{\mathfrak{c}[\mathfrak{b}\mathfrak{a}]} = F_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}\,; \quad \Pi_{\mathfrak{c}\mathfrak{b}\mathfrak{a}} = \Pi^{\mathfrak{b}}_{\mathfrak{c}\mathfrak{b}}\, f_{\mathfrak{a}\mathfrak{b}} \quad \ldots\ldots\ldots \quad (21)$$

Es folgt

$$S_{[\mathfrak{c}\mathfrak{b}\mathfrak{a}]} = \Pi_{[\mathfrak{c}\mathfrak{b}\mathfrak{a}]} = F_{[\mathfrak{c}\mathfrak{b}\mathfrak{a}]} \quad \ldots\ldots\ldots \quad (22)$$

Ist der Zusammenhang symmetrisch, so folgt aus (21), für den Fall dass $F_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}$ verschwindet, dass $\Pi_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}$ in allen Indizes symmetrisch ist,

$$\Pi_{\mathfrak{c}\mathfrak{b}\mathfrak{a}} = \Pi_{(\mathfrak{c}\mathfrak{b}\mathfrak{a})} \quad \ldots\ldots\ldots \quad (23)$$

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[1]) $\overset{h}{=}$ bedeutet: Gleichheit nur gültig für holonome Berugssysteme.

Aus (21) leitet sich noch ab

$$-\Pi^{\mathfrak{a}}_{\mathfrak{c}\mathfrak{b}}\, x_{\mathfrak{a}} = Q_{\mathfrak{b}\mathfrak{c}} - f_{\mathfrak{b}\mathfrak{c}} + F_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}\, x^{\mathfrak{a}} \quad \ldots \ldots \quad (24)$$

Sodass sich hier aus den $\Pi^{\mathfrak{a}}_{\mathfrak{c}\mathfrak{b}}$ noch eine dritte Grösse der Valenz zwei mit der Gradzahl 0 ableiten lässt, die sich aber in $Q_{\mathfrak{b}\mathfrak{a}}$, $f_{\mathfrak{b}\mathfrak{a}}$ und $F_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}$ ausdrücken lässt. Ueberschiebung von (20) mit $f^{\mathfrak{b}\mathfrak{a}}$ ergibt

$$-2\,\Pi^{\mathfrak{a}}_{\mathfrak{c}\mathfrak{a}} = F_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}\, f^{\mathfrak{b}\mathfrak{a}} = F_{\mathfrak{c}\,.\,\mathfrak{a}}^{\;.\,\mathfrak{a}} \quad \ldots \ldots \quad (25)$$

Festlegung eine Zusammenhanges mit Hilfe einer Doppelblattes.

Ein Affinor $B^{\varkappa}_{\lambda}$ vom Range p in einer E_n, $n = 2p$, der der Gleichung

$$B^{\varkappa}_{\varrho}\, B^{\varrho}_{\lambda} = B^{\varkappa}_{\lambda} \quad \ldots \ldots \quad (26)$$

genügt, bestimmt in dieser E_n eindeutig ein *Doppelblatt*, d. i. ein System von zwei E_p, die keine Richtung gemeinsam haben und umgekehrt ist $B^{\varkappa}_{\lambda}$ durch das Doppelblatt eindeutig bestimmt. In derselben Weise legt ein Kontaktaffinorfeld $B^{\mathfrak{a}}_{\mathfrak{b}}$ mit der Gradzahl Null vom Range $n+1$, das der Gleichung

$$B^{\mathfrak{a}}_{\mathfrak{c}}\, B^{\mathfrak{c}}_{\mathfrak{b}} = B^{\mathfrak{a}}_{\mathfrak{b}} \quad \ldots \ldots \quad (27)$$

genügt, in jedem Lokalraum ein Gebilde fest, das wir ebenfalls *Doppelblatt* nennen wollen. Für den Affinor $C^{\mathfrak{a}}_{\mathfrak{b}} = A^{\mathfrak{a}}_{\mathfrak{b}} - B^{\mathfrak{a}}_{\mathfrak{b}}$ gilt offenbar

$$C^{\mathfrak{a}}_{\mathfrak{c}}\, C^{\mathfrak{c}}_{\mathfrak{b}} = C^{\mathfrak{a}}_{\mathfrak{b}}; \quad B^{\mathfrak{a}}_{\mathfrak{c}}\, C^{\mathfrak{c}}_{\mathfrak{b}} = 0 \quad \ldots \ldots \quad (28)$$

Das Doppelblatt heisst involutorisch i. b. auf $f_{\mathfrak{b}\mathfrak{a}}$ wenn

$$B^{\mathfrak{b}\mathfrak{c}}_{\mathfrak{b}\mathfrak{a}}\, f_{\mathfrak{b}\mathfrak{c}} = 0; \quad C^{\mathfrak{b}\mathfrak{c}}_{\mathfrak{b}\mathfrak{a}}\, f_{\mathfrak{b}\mathfrak{c}} = 0 \quad \ldots \ldots \quad (29)$$

ist, oder, anders geschrieben

$$f_{\mathfrak{b}\mathfrak{a}} = B^{\mathfrak{b}}_{\mathfrak{b}}\, C^{\mathfrak{c}}_{\mathfrak{a}}\, f_{\mathfrak{b}\mathfrak{c}} + C^{\mathfrak{b}}_{\mathfrak{b}}\, B^{\mathfrak{c}}_{\mathfrak{a}}\, f_{\mathfrak{b}\mathfrak{c}}, \quad \ldots \ldots \quad (30)$$

woraus hervorgeht, dass

$$\left.\begin{aligned} B^{\mathfrak{e}}_{\mathfrak{b}}\, f_{\mathfrak{e}\mathfrak{a}} &= B^{\mathfrak{e}}_{\mathfrak{b}}\, C^{\mathfrak{b}}_{\mathfrak{a}}\, f_{\mathfrak{e}\mathfrak{b}} \\ C^{\mathfrak{e}}_{\mathfrak{b}}\, f_{\mathfrak{e}\mathfrak{a}} &= C^{\mathfrak{e}}_{\mathfrak{b}}\, B^{\mathfrak{b}}_{\mathfrak{a}}\, f_{\mathfrak{e}\mathfrak{b}} \end{aligned}\right\} \quad \ldots \ldots \quad (31)$$

ist. Aehnliche Gleichungen gelten für $f^{\mathfrak{a}\mathfrak{b}}$, da

$$B^{\mathfrak{b}}_{\mathfrak{c}}\, f^{\mathfrak{c}\mathfrak{a}}\, f_{\mathfrak{b}\mathfrak{b}} = B^{\mathfrak{b}}_{\mathfrak{c}}\, f^{\mathfrak{c}\mathfrak{a}}\, C^{\mathfrak{e}}_{\mathfrak{b}}\, B^{\mathfrak{f}}_{\mathfrak{b}}\, f_{\mathfrak{e}\mathfrak{f}} = f^{\mathfrak{c}\mathfrak{a}}\, C^{\mathfrak{e}}_{\mathfrak{b}}\, f_{\mathfrak{e}\mathfrak{c}} = -C^{\mathfrak{a}}_{\mathfrak{b}} \quad \ldots \quad (32)$$

ist, sodass (29) gleichwertig ist mit

$$C^{\mathfrak{a}\mathfrak{b}}_{\mathfrak{c}\mathfrak{b}}\, f^{\mathfrak{c}\mathfrak{b}} = 0; \quad B^{\mathfrak{a}\mathfrak{b}}_{\mathfrak{c}\mathfrak{b}}\, f^{\mathfrak{c}\mathfrak{b}} = 0 \quad \ldots \ldots \quad (33)$$

Wir sagen, dass eine Grösse mit einem Index im B- bzw. C-Gebiet liegt, wenn sie sich bei Ueberschiebung mit B bzw. C über diesen Index nicht ändert. Ueberschiebung mit C bzw. B über denselben Index erzeugt dann Null.

Es soll jetzt der Satz bewiesen werden:

Jeder lineare Zusammenhang in K_{2n-1} ist eindeutig bestimmt durch Angabe der folgenden Grössen:

1. *Der kovariante Differentialquotient von $f_{\mathfrak{b}\mathfrak{a}}$*

$$\nabla_{\mathfrak{c}} f_{\mathfrak{b}\mathfrak{a}} = F_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}. \qquad (34)$$

2. *Der kovariante Differentialquotient der zu irgend einem beliebigen in bezug auf $f_{\mathfrak{b}\mathfrak{a}}$ involutorischen Doppelblatt gehörigen Grösse $B_{\mathfrak{b}}^{\mathfrak{a}}$*

$$\nabla_{\mathfrak{c}} B_{\mathfrak{b}}^{\mathfrak{a}} = -\nabla_{\mathfrak{c}} C_{\mathfrak{b}}^{\mathfrak{a}} = E_{\mathfrak{c}\mathfrak{b}}^{\cdot\cdot\mathfrak{a}}. \qquad (35)$$

3. *Die zwei folgenden Teile von $S_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}$:*

$$\left.\begin{aligned} S_{\mathfrak{f}\mathfrak{e}\mathfrak{d}}\, B_{\mathfrak{c}}^{\mathfrak{f}}\, C_{\mathfrak{b}\mathfrak{a}}^{\mathfrak{e}\mathfrak{d}} &= T_{\mathfrak{c}\mathfrak{b}\mathfrak{a}} \\ S_{\mathfrak{f}\mathfrak{e}\mathfrak{d}}\, C_{\mathfrak{c}}^{\mathfrak{f}}\, B_{\mathfrak{b}\mathfrak{a}}^{\mathfrak{e}\mathfrak{d}} &= U_{\mathfrak{c}\mathfrak{b}\mathfrak{a}} \end{aligned}\right\} \qquad (36)$$

Beweis. $F_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}$ hat natürlich der Forderung $F_{\mathfrak{c}(\mathfrak{b}\mathfrak{a})} = 0$ zu genügen, $T_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}$ muss mit $\mathfrak{a}$ und $\mathfrak{b}$ im C-Gebiet und mit $\mathfrak{c}$ im B-Gebiet liegen und ebenso $U_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}$ mit $\mathfrak{a}$ und $\mathfrak{b}$ im B-Gebiet und mit $\mathfrak{c}$ im C-Gebiet. Weiter ergeben sich durch Differentiation von (29) folgende Beziehungen zwischen $F_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}$ und $E_{\mathfrak{c}\mathfrak{b}}^{\cdot\cdot\mathfrak{a}}$

$$\left.\begin{aligned} F_{\mathfrak{c}\mathfrak{e}\mathfrak{d}}\, B_{\mathfrak{b}\mathfrak{a}}^{\mathfrak{e}\mathfrak{d}} &= E_{\mathfrak{c}\mathfrak{b}\mathfrak{d}}\, B_{\mathfrak{a}}^{\mathfrak{d}} - E_{\mathfrak{c}\mathfrak{a}\mathfrak{d}}\, B_{\mathfrak{b}}^{\mathfrak{d}} \\ F_{\mathfrak{c}\mathfrak{e}\mathfrak{d}}\, C_{\mathfrak{b}\mathfrak{a}}^{\mathfrak{e}\mathfrak{d}} &= -E_{\mathfrak{c}\mathfrak{b}\mathfrak{d}}\, C_{\mathfrak{a}}^{\mathfrak{d}} + E_{\mathfrak{c}\mathfrak{a}\mathfrak{d}}\, C_{\mathfrak{b}}^{\mathfrak{d}} \end{aligned}\right\} \qquad (37)$$

woraus folgt

$$\left.\begin{aligned} &E_{\mathfrak{c}\mathfrak{e}\mathfrak{d}}\, C_{\mathfrak{b}}^{\mathfrak{e}}\, B_{\mathfrak{a}}^{\mathfrak{d}} = 0 \qquad E_{\mathfrak{c}\mathfrak{e}\mathfrak{d}}\, B_{\mathfrak{b}}^{\mathfrak{e}}\, C_{\mathfrak{a}}^{\mathfrak{d}} = 0 \\ &F_{\mathfrak{c}\mathfrak{e}\mathfrak{d}}\,(B_{\mathfrak{b}\mathfrak{a}}^{\mathfrak{e}\mathfrak{d}} - C_{\mathfrak{b}\mathfrak{a}}^{\mathfrak{e}\mathfrak{d}}) = 2\, E_{\mathfrak{c}[\mathfrak{b}\mathfrak{a}]} \end{aligned}\right\} \qquad (38)$$

Wird diesen Forderungen genügt, so bleiben für $F_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}$ noch $^1/_2\, N^3 - {^1/_2}\, N^2$ für $E_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}$, wenn $F_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}$ schon festgelegt ist, noch $^1/_4\, N^3 + {^1/_2}\, N^2$, für $T_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}$ noch $^1/_8\, N^3$ und für $U_{\mathfrak{c}\mathfrak{b}\mathfrak{a}}$ noch $^1/_8\, N^3$ freie Bestimmungszahlen ($N = 2n+2$), im Ganzen N^3 und es gibt auch gerade N^3 Unbekannte, die $\Pi_{\mathfrak{c}\mathfrak{b}}^{\mathfrak{a}}$. Aus $f_{\mathfrak{b}\mathfrak{a}}$ und $B_{\mathfrak{b}}^{\mathfrak{a}}$ lässt sich ein Tensor

$$G_{\mathfrak{b}\mathfrak{a}} = 2 f_{(\mathfrak{b}|\mathfrak{d}|}\, B_{\mathfrak{a})}^{\mathfrak{d}} \qquad (39)$$

bilden, dessen Rang infolge der involutorischen Lage von $B_{\mathfrak{b}}^{\mathfrak{a}}$ gleich $2n+2$ ist. Differentiation dieser Gleichung ergibt

$$\left.\begin{aligned}\nabla_{\mathfrak{c}} G_{\mathfrak{b}\mathfrak{a}} = F_{\mathfrak{c}\mathfrak{b}\mathfrak{d}} B_{\mathfrak{a}}^{\mathfrak{d}} + E_{\mathfrak{c}\mathfrak{b}\mathfrak{a}} \\ + F_{\mathfrak{c}\mathfrak{a}\mathfrak{d}} B_{\mathfrak{b}}^{\mathfrak{d}} + E_{\mathfrak{c}\mathfrak{a}\mathfrak{b}}\end{aligned}\right\} \quad \ldots \ldots \quad (40)$$

(35) lautet ausgeschrieben

$$E_{\mathfrak{c}\mathfrak{b}}^{\cdot\cdot\mathfrak{a}} - \partial_{\mathfrak{c}} B_{\mathfrak{b}}^{\mathfrak{a}} = -\Pi_{\mathfrak{c}\mathfrak{b}}^{\mathfrak{d}} B_{\mathfrak{d}}^{\mathfrak{a}} + \Pi_{\mathfrak{c}\mathfrak{d}}^{\mathfrak{a}} B_{\mathfrak{b}}^{\mathfrak{d}} \quad \ldots \ldots \quad (41)$$

woraus hervorgeht, dass

$$\left.\begin{aligned}-C_{\mathfrak{b}}^{\mathfrak{d}} (E_{\mathfrak{c}\mathfrak{d}}^{\cdot\cdot\mathfrak{a}} - \partial_{\mathfrak{c}} B_{\mathfrak{d}}^{\mathfrak{a}}) = \Pi_{\mathfrak{c}\mathfrak{b}}^{\mathfrak{e}} B_{\mathfrak{e}}^{\mathfrak{a}} C_{\mathfrak{b}}^{\mathfrak{b}} = \Pi_{\mathfrak{c}\mathfrak{b}\mathfrak{f}} C_{\mathfrak{b}}^{\mathfrak{b}} f^{\mathfrak{f}\mathfrak{e}} B_{\mathfrak{e}}^{\mathfrak{a}} \\ = \Pi_{\mathfrak{c}\mathfrak{b}\mathfrak{g}} C_{\mathfrak{b}\mathfrak{f}}^{\mathfrak{b}\mathfrak{g}} f^{\mathfrak{f}\mathfrak{a}}\end{aligned}\right\} \quad \ldots \quad (42)$$

und

$$C_{\mathfrak{e}}^{\mathfrak{a}} (E_{\mathfrak{c}\mathfrak{b}}^{\cdot\cdot\mathfrak{e}} - \partial_{\mathfrak{c}} B_{\mathfrak{b}}^{\mathfrak{e}}) = \Pi_{\mathfrak{c}\mathfrak{b}}^{\mathfrak{e}} C_{\mathfrak{e}}^{\mathfrak{a}} B_{\mathfrak{b}}^{\mathfrak{b}} = \Pi_{\mathfrak{c}\mathfrak{b}\mathfrak{g}} B_{\mathfrak{b}\mathfrak{f}}^{\mathfrak{b}\mathfrak{g}} f^{\mathfrak{f}\mathfrak{a}} \quad \ldots \quad (43)$$

sodass

$$\left.\begin{aligned}\Pi_{\mathfrak{c}\mathfrak{e}\mathfrak{d}} C_{\mathfrak{b}\mathfrak{a}}^{\mathfrak{e}\mathfrak{d}} = -f_{\mathfrak{a}\mathfrak{e}} C_{\mathfrak{b}}^{\mathfrak{d}} (E_{\mathfrak{c}\mathfrak{d}}^{\cdot\cdot\mathfrak{e}} - \partial_{\mathfrak{c}} B_{\mathfrak{d}}^{\mathfrak{e}}) \\ \Pi_{\mathfrak{c}\mathfrak{e}\mathfrak{d}} B_{\mathfrak{b}\mathfrak{a}}^{\mathfrak{e}\mathfrak{d}} = f_{\mathfrak{a}\mathfrak{d}} C_{\mathfrak{e}}^{\mathfrak{d}} (E_{\mathfrak{c}\mathfrak{b}}^{\cdot\cdot\mathfrak{e}} - \partial_{\mathfrak{c}} B_{\mathfrak{b}}^{\mathfrak{e}}).\end{aligned}\right\} \quad \ldots \ldots \quad (44)$$

ist. Es sind also jetzt nur noch $\Pi_{\mathfrak{c}\mathfrak{e}\mathfrak{d}} B_{\mathfrak{b}}^{\mathfrak{e}} C_{\mathfrak{a}}^{\mathfrak{d}}$ und $\Pi_{\mathfrak{c}\mathfrak{e}\mathfrak{d}} C_{\mathfrak{b}}^{\mathfrak{e}} B_{\mathfrak{a}}^{\mathfrak{d}}$ zu bestimmen.
Aus (36) folgt:

$$\Pi_{\mathfrak{f}\mathfrak{e}\mathfrak{d}} B_{\mathfrak{c}}^{\mathfrak{f}} C_{\mathfrak{b}\mathfrak{a}}^{\mathfrak{e}\mathfrak{d}} - \Pi_{\mathfrak{e}\mathfrak{f}\mathfrak{d}} B_{\mathfrak{c}}^{\mathfrak{f}} C_{\mathfrak{b}\mathfrak{a}}^{\mathfrak{e}\mathfrak{d}} = 2T_{\mathfrak{c}\mathfrak{b}\mathfrak{a}} \quad \ldots \ldots \quad (45)$$

oder

$$\Pi_{\mathfrak{f}\mathfrak{e}\mathfrak{d}} C_{\mathfrak{c}}^{\mathfrak{f}} B_{\mathfrak{b}}^{\mathfrak{e}} C_{\mathfrak{a}}^{\mathfrak{d}} = -2T_{\mathfrak{b}\mathfrak{c}\mathfrak{a}} + \Pi_{\mathfrak{f}\mathfrak{e}\mathfrak{d}} B_{\mathfrak{b}}^{\mathfrak{f}} C_{\mathfrak{c}\mathfrak{a}}^{\mathfrak{e}\mathfrak{d}} \quad \ldots \ldots \quad (46)$$

und unter Berücksichtigung von (21)

$$\Pi_{\mathfrak{f}\mathfrak{e}\mathfrak{d}} C_{\mathfrak{c}\mathfrak{a}}^{\mathfrak{f}\mathfrak{e}} B_{\mathfrak{b}}^{\mathfrak{d}} = \Pi_{\mathfrak{f}\mathfrak{e}\mathfrak{d}} C_{\mathfrak{c}}^{\mathfrak{f}} B_{\mathfrak{b}}^{\mathfrak{e}} C_{\mathfrak{a}}^{\mathfrak{d}} + F_{\mathfrak{f}\mathfrak{e}\mathfrak{d}} C_{\mathfrak{c}\mathfrak{a}}^{\mathfrak{f}\mathfrak{e}} B_{\mathfrak{b}}^{\mathfrak{d}} \quad \ldots \quad (47)$$

In derselben Weise ergeben sich aus (36) und (21) $\Pi_{\mathfrak{f}\mathfrak{e}\mathfrak{d}} B_{\mathfrak{c}}^{\mathfrak{f}} C_{\mathfrak{b}}^{\mathfrak{e}} B_{\mathfrak{a}}^{\mathfrak{d}}$ und $\Pi_{\mathfrak{f}\mathfrak{e}\mathfrak{d}} B_{\mathfrak{c}\mathfrak{b}}^{\mathfrak{f}\mathfrak{e}} C_{\mathfrak{a}}^{\mathfrak{d}}$, und durch Addition erhält man dann die $\Pi_{\mathfrak{c}\mathfrak{b}}^{\mathfrak{a}}$ in bekannte Grössen ausgedrückt.

Umdeutung der Resultate für die linearen Uebertragungen in einer X_n mit geradem n.

In der X_n ($n =$ gerade) mit einem Fundamentalbivektor $f_{\lambda\varkappa}$ vom Range

n und einem Doppelblattfeld $B_\lambda^{\cdot\varkappa}$ in involutorischen Lage i. b. auf $f_{\lambda\varkappa}$ lassen sich ähnliche Betrachtungen aufstellen. Aus $f_{\lambda\varkappa}$ und $B_\lambda^{\cdot\varkappa}$ lässt sich ein Fundamentaltensor $g_{\lambda\varkappa}$ vom Range n ableiten. Jede lineare Uebertragung ist festgelegt durch $\nabla_\mu f_{\lambda\varkappa}$, $\nabla_\mu B_\lambda^{\cdot\varkappa}$, $S_{\tau\sigma\varrho} B_\mu^{\cdot\tau} C_{\lambda\varkappa}^{\sigma\varrho}$ und $S_{\tau\sigma\varrho} C_\mu^{\cdot\tau} B_{\lambda\varkappa}^{\sigma\varrho}$. Für den Fall, dass alle diese Grössen verschwinden, ergibt sich der Satz:

Es gibt in einer X_n eine und nur eine lineare Uebertragung, die einen Fundamentalbivektor und ein in bezug auf diesen Bivektor involutorisch gelegenes Doppelblattfeld invariant lässt und ausserdem infinitesimale Parallelogramme zulässt in jeder 2-Richtung, die mit den beiden p-Richtungen des Doppelblattes je eine Richtung gemeinsam hat. Diese Uebertragung lässt einen Fundamentaltensor invariant und ist also metrisch. Sie ist im allgemeinen nicht symmetrisch, genügt aber der Gleichung $S_{[\mu\lambda\varkappa]} = 0$.

Astronomy. — *Mittlere Lichtkurven von langperiodischen Veränderlichen.* XXXI. *Y Cassiopeiae.* Von A. A. NIJLAND †.

(Communicated at the meeting of May 28, 1938)[1]).

Die Beobachtungen wurden alle in *R* observiert. Spektrum M6e—8 (HA 79). Gesamtzahl der Beobachtungen 720 (von 2417249 bis 2428335). Sechs stark abweichende Schätzungen (2421957, 2412, 3111, 3419, 5733 und 7617), in der Figur 1 eingeklämmert, wurden verworfen. In 41 in der Figur 1 mit V bezeichneten Fällen war der Stern unsichtbar. Es bleiben dann 673 Beobachtungen für die Diskussion übrig.

Karte: HAGEN, *Atlas Stell. var. Series VI.*

Die Tabelle I gibt eine Uebersicht der benützten Vergleichsterne. Die

TABELLE I. Vergleichsterne.

*	BD	HAGEN	St.	HA 57	Grenze	*H*
A	$+54^{\circ}3092$	—	42.7	—	—	$9^{m}.11$
a	54.3099	—	35.9	$9^{m}.87$	—	9.87
b	54.3102	—	33.3	—	—	10.17
c	—	28	28.3	—	—	10.75
d	—	40	22.4	—	—	11.42
e	—	60	16.3	—	—	12.13
f	—	65	11.7	—	—	12.67
g	—	70	6.9	—	$13^{m}.43$	13.22
γ	—	—	5.6	—	13.67	13.37
h	—	81	0.0	—	14.01	14.01

Sterne *g*, γ und *h* wurden 8-, bezw. 6- und 94-mal an die Grenze von *R*

[1]) Die von NIJLAND hinterlassenen Beobachtungen der noch nicht bearbeiteten langperiodischen Veränderlichen sind unter Direktion von Dr. VAN DER BILT und Dr. MINNAERT durch D. VAN SUYLEN, der als Gehilfe NIJLAND's auch die früheren Sterne behandelt hat, in genau derselben Weise reduziert worden. Die Resultate werden in derselben Weise wie bei den früheren Sternen in diesen Proceedings veröffentlicht werden. (Vgl. diese Proceedings, Vol. **40**, S. 391.)

angeschlossen; die sich hieraus ergebenden Helligkeiten sind: $g = 13^m.43$, $\gamma = 13^m.67$, $h = 14^m.01$. Der Stufenwert ist $0^m.115$.

Es liegen 39 Schätzungen der Farbe vor. Aus der Tabelle II*a* scheint hervorzugehen, dasz sich die Farbe in den Jahren 1906—1918 um etwa 2^c vertieft hat. Nach der Tabelle II*b* scheint sich die Farbe von $4^c.85$ bei $9^m.65$ bis $6^c.17$ bei $10^m.85$ zu vertiefen. Das allgemeine Mittel ist $5^c.34$.

TABELLEN IIa und II*b*. Farbenschätzungen.

Zeitraum	*n*	Farbe	Grösse	*n*	Farbe
		c	m		c
2417578—2418010	10	4.25	9.65	10	4.85
8024— 8894	10	5.10	10.11	10	5.15
2419272—2421862	10	6.30	10.36	10	5.20
1871— 8049	9	5.67	10 85	9	6.17
	39			39	

Die Figur 1 enthält die Beobachtungen. Die Reihe der Abweichungen (Beobachtung minus Kurve) zeigt 249 Plus-, 204 Minuszeichen, 220 Nullwerte, 222 Zeichenfolgen, 230 Zeichenwechsel. Das Mittel der absoluten Werte der Abweichungen ist $0^m.083$.

Ein Einfluss des Mondscheines auf die Helligkeitsschätzung ist nicht bemerkbar. Es verteilen sich auf 118 bei Mondschein angestellte Beobachtungen die Abweichungen wie folgt: 35 Plus-, 42 Minuszeichen und 41 Nullwerte.

Die Tabelle III enthält die aus der Kurve abgelesenen Epochen der Minima *m* und der Maxima *M*. Die Spalte *R* wurde mit den einfachen Elementen:

$$2422472^d + 417^d.8\ E \quad \text{(für die Minima)}$$

und

$$2422646 \;\; + 417\ .8\ E \quad \text{(für die Maxima)}$$

gerechnet.

Die übrigbleibenden *B*—*R* sind sehr gross und zeigen in den beiden Spalten einen systematischen Charakter, so dass man die Abweichungen für die Maxima und die Minima vereinigen kann (Figur 2). Da es aber nicht gelungen ist, hier mit einfachen Mitteln viel zu verbessern, habe ich mich mit der einfachen Formel *R* zufriedengestellt.

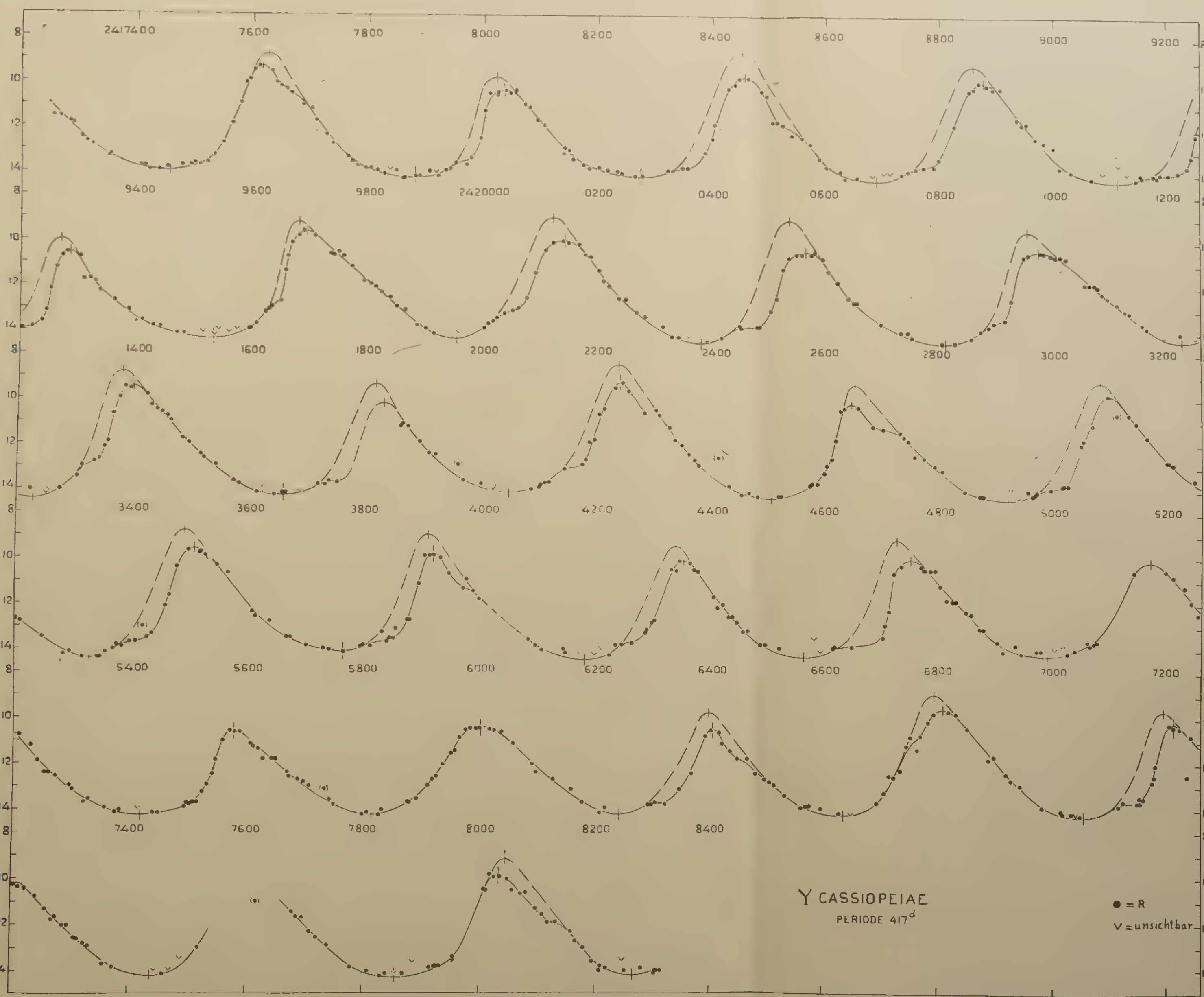
SCHNELLER's Katalog für 1937 gibt den Periodenwert $407^d.9$.

Die extremen Werte des Lichtwechsels sind:

$$\left.\begin{array}{ll}\text{Minimum:} & 14^m.27 \pm 0^m.02 \\ \text{Maximum:} & 9\ .98 \pm 0\ \ .08\end{array}\right\}\text{(m.F.)}$$

Die Amplitude beträgt somit $4^m.29$.

A. A. NIJLAND †: MITTLERE LICHTKURVEN VON LANGPERIODISCHEN VERÄNDERLICHEN. XXXI. Y CASSIOPEIAE.



Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLI, 1938.

TABELLE III.

E	Minima m				Maxima M			
	B	v	R	B—R	B	v	R	B—R
—12	241 7455	m 13.9	7458	— 3	241 7615	m 9.3	7632	—17
11	7880	14.2	7876	— 4	8035	10.3	8050	—15
10	8272	14.2	8294	—22	8456	9.7	8468	—12
9	8689	14.3	8712	—23	8877	9.9	8886	— 9
8	9112	14.4	9130	—18	9285	10.5	9304	—19
7	9532	14.4	9547	—15	9693	9.6	9721	—28
6	9954	14.3	9965	—11	242 0142	10.0	0139	+ 3
5	242 0380	14.5	0383	— 3	0566	10.5	0557	+ 9
4	0812	14.5	0801	+11	0973	10.4	0975	— 2
3	1228	14.4	1219	+ 9	1397	9.5	1393	+ 4
2	1656	14.2	1636	+20	[1830]	[10.1]	—	—
— 1	2046	14.1	2054	— 8	2239	9.3	2228	+11
0	2508	14.3	2472	+36	2647	10.2	2646	+ 1
+ 1	2921	14.4	2890	+31	3099	9.8	3064	+35
2	3329	14.4	3308	+21	3505	9.6	3482	+23
3	3763	14.1	3725	+38	3918	9.8	3899	+19
4	4178	14.4	4143	+35	4352	10.1	4317	+35
5	4566	14.3	4561	+ 5	4751	10.0	4735	+16
6	4991	14.3	4979	+12	5170	10.1	5153	+17
7	5419	14.2	5397	+22	5578	10.5	5571	+ 7
8	5814	14.2	5814	0	6001	10.4	5988	+13
9	6240	14.2	6232	+ 8	6405	10.5	6406	— 1
10	6635	14.2	6650	—15	6809	9.6	6824	—15
11	7056	14.3	7068	—12	7213	10.2	7242	—29
12	7436	14.2	7486	—50	[7608]	[9.6]	—	—
13	7856	14.3	7903	—47	8033	9.8	8077	—44
+14	8265	14.2	8321	—56				
		14.27				9.98		

TABELLE IV. Die mittlere Kurve.

Phase	v	Phase	v	Phase	v	Phase	v	Phase	v
d	m	d	m	d	m	d	m	d	m
—140	12.48	—40	14.12	+ 60	13.78	+160	10.15	+260	12.02
—130	12.69	—30	14.18	+ 70	13.69	+170	9.99	+270	12.27
—120	12.91	—20	14.22	+ 80	13.58	+180	10.01	+280	12.53
—110	13.12	—10	14.27	+ 90	13 43	+190	10.20	+290	12.75
—100	13.32	0	14.27	+100	13.23	+200	10.48	+300	13.00
— 90	13.50	+10	14.25	+110	12.95	+210	10.75	+310	13.22
— 80	13.67	+20	14.20	+120	12.52	+220	11.02		
— 70	13.81	+30	14.13	+130	11.94	+230	11.28		
— 60	13.94	+40	14.05	+140	11.25	+240	11.53		
— 50	14.05	+50	13.91	+150	10.53	+250	11.78		

Auch für Y Cassiopeiae wurde wieder der mitllere Verlauf der Lichtkurve in der Nähe der beiden Hauptphasen durch Ablesung der Helligkeit für je 10^d abgeleitet. Die sehr unsichere Maxima 2421830 und 2427608 blieben hierbei unberücksichtigt. Die beiden Teilkurven schliessen sich befriedigend an einander an (s. die Figur 3) und geben zusammen den Verlauf der mittleren Kurve B (Tabelle IV). Wird auch bei Y Cassiopeiae in der üblichen Weise die Kurve von der Störung befreit, so entstehen die ungestörten Maxima, welche (Tabelle V) mit den einfachen Elementen R:

$$2422634^d + 417^d.8\ E$$

verglichen wurden.

Die maximale „ungestörte" Helligkeit ist:

$$V = 9^m.18 \pm 0^m.08\ (\text{m.F.}).$$

Die Teilkurve A der ungestörten Maxima (Fig. 3) schliesst sich

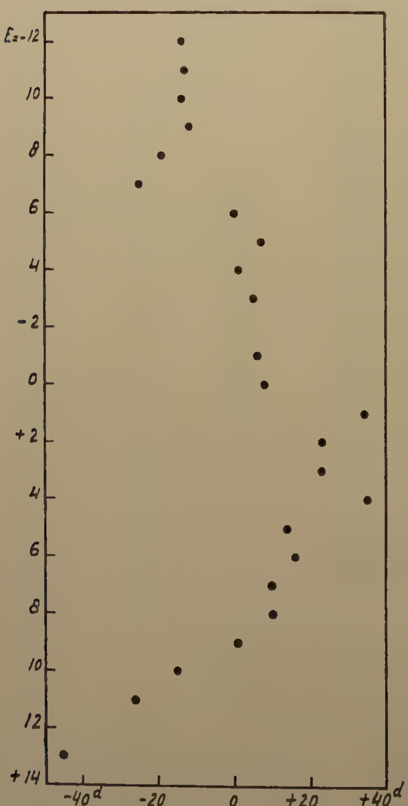


Fig. 2.

TABELLE V. Ungestörte Maxima.

E	B	V	R	B—R	E	B	V	R	B—R
—12	241 7626	m 8.8	7620	+ 6	+ 1	242 3082	m 9.2	3052	+30
11	8021	9.7	8038	—17	2	3488	8.8	3470	+18
10	8452	8.6	8456	— 4	3	3908	9.0	3887	+21
9	8860	9.2	8874	—14	4	4338	9.5	4305	+33
8	9270	10.0	9292	—22	5	4728	9.1	4723	+ 5
7	9680	9.2	9709	—29	6	—	—	—	—
6	242 0120	9.0	0127	— 7	7	—	—	—	—
5	0536	9.1	0545	— 9	8	—	—	—	—
4	0954	9.5	0963	— 9	9	6398	9.7	6394	+ 4
3	1380	8.8	1381	— 1	10	6793	9.0	6812	—19
2	1816	9.3	1798	+18	11	7195	9.6	7230	—35
— 1	2236	8.5	2216	+20	12	—	—	—	—
0	2653	9.3	2634	+19	+13	8045	9.1	8065	—20

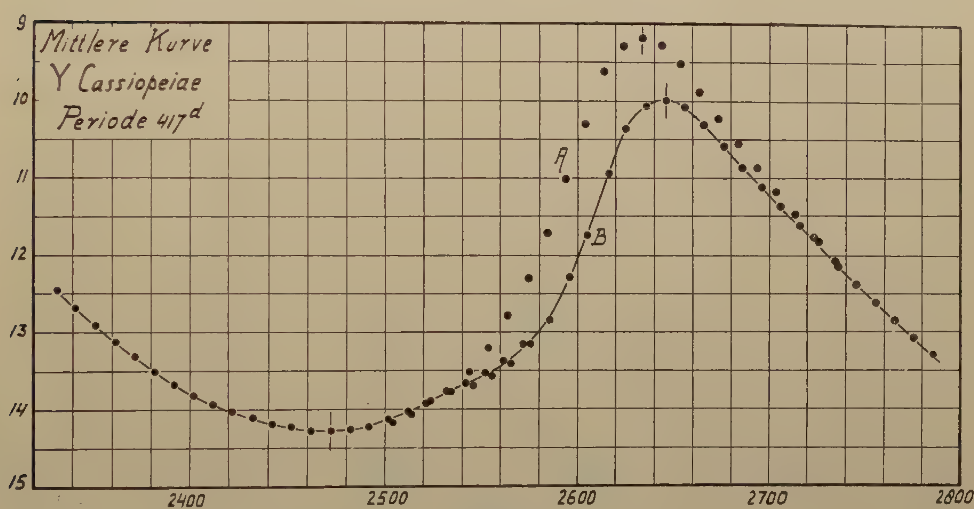


Fig. 3.

derjenigen der Minima vollständig an. Für die Schiefe der ungestörten Kurven findet man: $\frac{M-m}{P} = 0.388.$

Utrecht, September 1937.

Astronomy. — *Mittlere Lichtkurven von langperiodischen Veränderlichen.* XXXII. *R V Pegasi.* Von A. A. NIJLAND †.

(Communicated at the meeting of May 28, 1938.)

Instrumente *S* und *R*. Die Beobachtungen wurden alle auf *R* reduziert; die Reduktion *R*—*S* beträgt —$0^m.21$. Spektrum M6e (H.A. **79**). Gesamtzahl der Beobachtungen 648 (von 2417053 bis 2428313). Es wurden wieder, wie in allen früheren Mitteilungen, die in zwei Instrumenten angestellten Schätzungen nur einmal gezählt. Dreizehn stark abweichende Schätzungen, in der Figur 1 eingeklämmert, wurden verworfen. Der Stern wurde 83-mal als unsichtbar notiert, und so bleiben 552 Beobachtungen für die Reduktion übrig.

Die Tabelle I gibt eine Uebersicht der benutzten Vergleichsterne.

TABELLE I. Vergleichsterne.

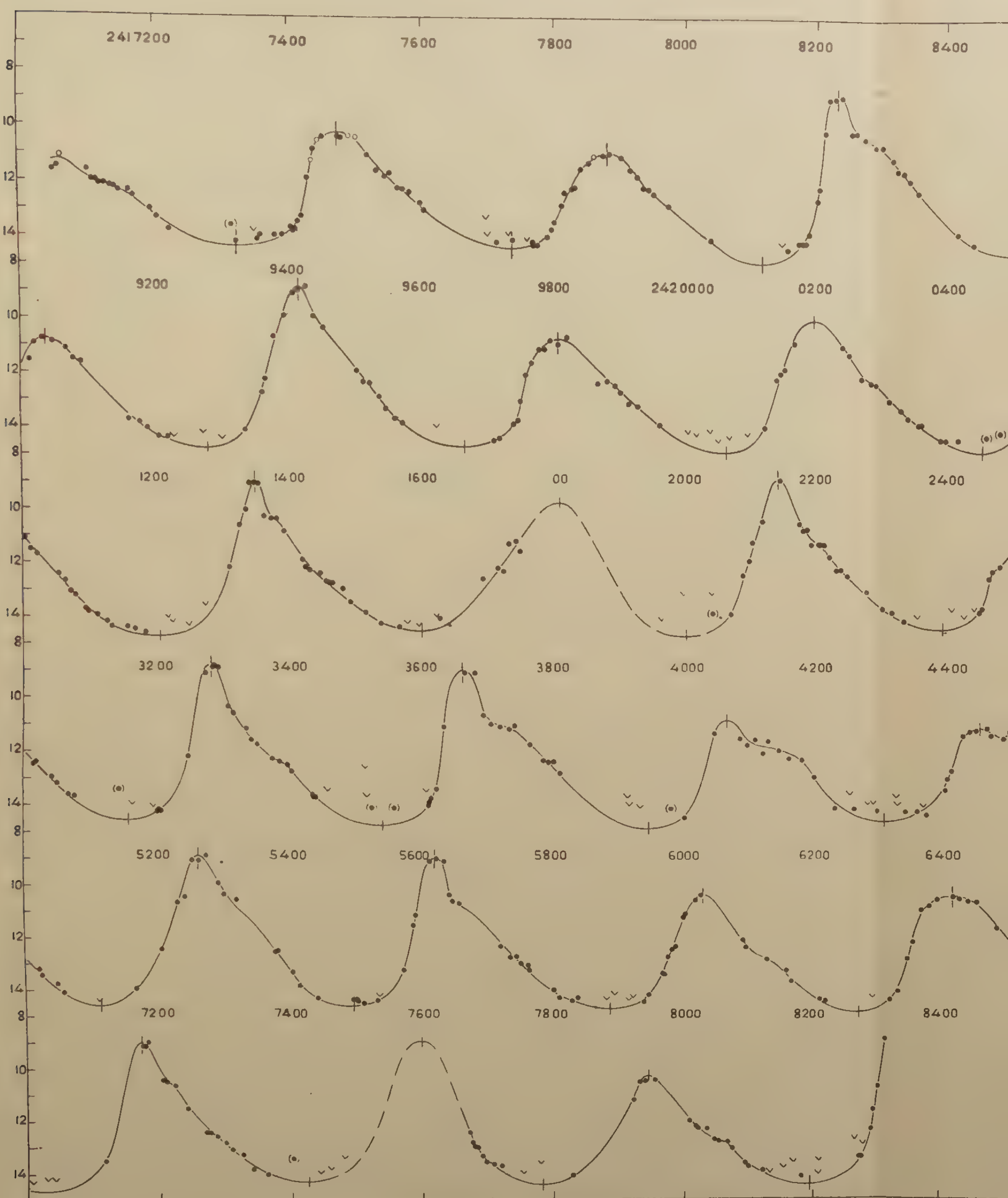
*	*BD*	* — v	St.	HA 57	HA 74	Grenze	*H*
b	+29.4653	—	41.0	$8^m.72$	$8^m.70$; $9^m.12$		$9^m.16$
x	+29.4655	—	34.8	9.35	9.20; 9.79		9.89
c	+29.4646	—	32.7	—			10.13
d	—	—	27.6	10.94			10.74
e	—	—	22.9	11.11			11.30
f	—	—	19.3	11.74			11.72
g	—	—	12.5	12.54		[$13^m.75$]	12.53
h	—	—$0^m 1^s$; —1′30″	7.1	—		13.54	13.17
k	—	+0 4; —0 30	0.0	—		13.94	14.00

Stern *g*, *h* und *k* wurden 8-, bezw. 38-, und 104-mal an die Grenze von *R* angeschlossen; die sich hieraus ergebenden Helligkeiten sind: $g = 13^m.75$, $h = 13^m.54$, $k = 13^m.94$ [1]). Der Stufenwert ist $0^m.118$.

Es liegen 7 Schätzungen der Farbe vor (Tabelle II*a* und II*b*). Das allgemeine Mittel ist $3^c.08$.

Die Figur 1 enthält die Beobachtungen, alle auf *R* reduziert. Die Reihe der Abweichungen (Beobachtung minus Kurve) zeigt 178 Plus-, 197

[1]) Der Wert für *g* ist sicher zu schwach.


2417200
7400
7600
7800
8000
8200
8400
9200
9400
9600
9800
2420000
0200
0400
1200
1400
1600
00
2000
2200
2400
3200
3400
3600
3800
4000
4200
4400
5200
5400
5600
5800
6000
6200
6400
7200
7400
7600
7800
8000
8200
8400
8
10
12
14

Minuszeichen, 177 Nullwerte, 185 Zeichenfolgen, 189 Zeichenwechsel. Das Mittel der absoluten Werte der Abweichungen ist $0^m.097$.

Ein Einfluss des Mondscheines auf die Helligkeitsschätzung ist nicht nachweisbar. Es verteilen sich auf 104 bei Mondschein angestellte Beobachtungen die Abweichungen wie folgt: 46 Plus-, 28 Minuszeichen, 30 Nullwerte.

TABELLE IIa und II*b*. Farbenschätzungen.

Zeitraum	*n*	Farbe	Grösse	*n*	Farbe
2417435—2418625	4	$2^c.50$	$8^m.86$	4	$3^c.50$
2418632—2424851	3	3.66	10.70	3	2.66
	7			7	

Die Tabelle III enthält die aus der Kurve abgelesenen Epochen der Minima *m* und der Maxima *M*. Die Spalte *R* wurde mit den einfachen Elementen:

$$2422382^d + 388^d.0\ E \text{ (für die Minima)}$$
$$2422518 \ \ + 388\ .0\ E \text{ (für die Maxima)}$$

gerechnet.

Da es nicht gelungen ist, hier mit einfachen Mitteln viel zu verbessern, habe ich mich mit den oben angegebenen Elementen zufrieden gestellt.

Für $\frac{M-m}{P}$ findet man 0.351.

SCHNELLER's Katalog für 1937 gibt den Periodenwert $388^d.5$.

Die extremen Werte des Lichtwechsels sind:

$$\left.\begin{array}{ll}\text{Minimum:} & 14^m.61 \pm 0^m.03 \\ \text{Maximum:} & 9\ .65 \pm 0\ .17\end{array}\right\} \text{(m.F.)}.$$

Die Amplitude beträgt somit $4^m.96$.

Es wurde wieder der mittlere Verlauf der Lichtkurve in der Umgebung der beiden Hauptphasen durch Ablesung der Helligkeit für je 10^d abgeleitet. Die beiden Teilkurven, die sich im Aufstieg etwas weniger genau als sonst an einander anschliessen (s. die Figur 2), geben zusammen den Verlauf der mittleren Kurve (Tabelle IV).

Die Streuung in der Nähe von 80^d erreicht die Werte:

	m	*M*	Mittel
im aufsteigenden Aste:	$0^m.298$	$0^m.928$	$0^m.613$
im absteigenden Aste :	0 .080	0 .567	0 .323
Mittel:	0 .189	0 .747	

Das Verhältnis der Streuungen $0^m.613$ und $0^m.323$ ist 1.90, das Ver-

TABELLE III.

E	Minima m				Maxima M			
	B	v	R	B—R	B	v	R	B—R
—13	241 7324	m 14.4	7338	—14	241 7474	m 10.2	7474	0
12	7737	14.4	7726	+11	7880	11.0	7862	+18
11	8115	15.0	8114	+ 1	8231	8.9	8250	—19
10	8496	14.6	8502	— 6	8638	8.8	8638	0
9	8889	14.7	8890	— 1	9036	10.8	9026	+10
8	9279	14.8	9278	+ 1	9416	8.9	9414	+ 2
7	9666	14.7	9666	0	9806	10.7	9802	+ 4
6	242 0060	14.9	0054	+ 6	0194	10.1	242 0190	+ 4
5	0454	14.9	0442	+12	0555	8.8	0578	—23
4	0842	14.6	0830	+12	0966	10.5	0966	0
3	1206	14.7	1218	—12	1348	9.0	1354	— 6
2	1600	14.5	1606	— 6	[1809]	[9.8]	—	—
— 1	[2000]	[14.7]	—	—	2139	8.9	2130	+ 9
0	2398	14.4	2382	+16	2530	8.7	2518	+12
+ 1	2766	14.5	2770	— 4	2906	8.9	2906	0
2	3158	14.5	3158	0	3281	8.8	3294	—13
3	3540	14.7	3546	— 6	3661	9.0	3682	—21
4	3944	14.8	3934	+10	4061	10.8	4070	— 9
5	4310	14.6	4322	—12	4455	11.1	4458	— 3
6	4696	14.5	4710	—14	4840	8.5	4846	— 6
7	5112	14.5	5098	+14	5261	8.9	5234	+27
8	5495	14.5	5486	+ 9	5618	9.0	5622	— 4
9	5885	14.6	5874	+11	6026	10.3	6010	+16
10	6270	14.8	6262	+ 8	6416	10.4	6398	+18
11	6659	14.6	6650	+ 9	6788	10.6	6786	+ 2
12	7031	14.7	7038	— 7	7173	9.0	7174	— 1
13	7425	14.3	7426	— 1	[7600]	—	—	—
14	7782	14.4	7814	—32	7944	10.2	7950	— 6
+15	8194	14.4	8202	— 8				
		14.61				9.65		

TABELLE IV. Die mittlere Kurve.

Phase	v	Phase	v	Phase	v	Phase	v	Phase	v
-140^d	$12^m.48$	-50^d	$14^m.33$	$+40^d$	$14^m.35$	$+130^d$	$9^m.71$	$+220^d$	$12^m.00$
-130	12.77	-40	14.42	$+50$	14.20	$+140$	9.70	$+230$	12.22
-120	13.02	-30	14.50	$+60$	13.96	$+150$	10.00	$+240$	12.46
-110	13.27	-20	14.56	$+70$	13.62	$+160$	10.34	$+250$	12.71
-100	13.50	-10	14.60	$+80$	13.15	$+170$	10.65	$+260$	12.94
-90	13.72	0	14.61	$+90$	12.51	$+180$	10.98	$+270$	13.20
-80	13.90	$+10$	14.57	$+100$	11 92	$+190$	11.25	$+280$	13.44
-70	14.06	$+20$	14.52	$+110$	10.96	$+200$	11.50		
-60	14.20	$+30$	14.45	$+120$	10.15	$+210$	11.76		

Fig. 2.

hältnis der durchschnittlichen Geschwindigkeiten des Lichtwechsels bei Auf- und Abstieg ist 1.85.

Utrecht. Februar 1938.

Astronomy. — *Mittlere Lichtkurven von langperiodischen Veränderlichen.* XXXIII. *R Z Pegasi.* Von A. A. NIJLAND †.

(Communicated at the meeting of May 28, 1938.)

Instrumente *S* und *R*. Die Beobachtungen wurden alle auf *R* reduziert; die Reduktion *R*—*S* beträgt —$0^{m}.15$. Spektrum Se (HA **79**). Gesamtzahl der Beobachtungen 235 (von 2423812 bis 2428335). Es wurden wieder, wie in allen früheren Mitteilungen, die in zwei Instrumenten angestellten Schätzungen nur einmal gezählt. Eine stark abweichende Schätzung (2427480), in der Figur 1 eingeklämmert, wurde verworfen, und so bleiben 234 Beobachtungen für die Diskussion übrig.

Die Tabelle I gibt eine Uebersicht der benutzten Vergleichsterne. Der Stufenwert ist $0^{m}.097$.

TABELLE I. Vergleichsterne.

*	HAGEN	BD	St.	HA 63	HA 74	*H*
a	10	+32°.4343	45.7	$8^{m}.52$	$8^{m}.60$	$8^{m}.55$
c	14	32.4340	38.4	9.14	9.07	9.27
d	20	32.4337	34.4	9.64	9.70	9.66
e	28	32.4334	32.0	9.94	9.83	9.89
f	38	32.4331	29.9	10.22	10.12	10.09
g	41	—	25.5	—	—	10.52
p	44	32.4336	18.8	11.10	11.16	11.17
q	51	—	11.6	11 84	11.95	11.87
t	56	—	5.6			12.45
u	59	—	0.0			13.00

Es liegen 12 Schätzungen der Farbe vor (Tabelle II*a* und II*b*). Das allgemeine Mittel ist $4^{c}.71$.

Die Figur 1 enthält die Beobachtungen, alle auf *R* reduziert. Die Reihe der Abweichungen (Beobachtung minus Kurve) zeigt 101 Plus-, 85 Minuszeichen, 48 Nullwerte, 107 Zeichenfolgen, 78 Zeichenwechsel. Das Mittel der absoluten Werte der Abweichungen ist $0^{m}.137$.

Ein Einfluss des Mondscheines auf die Helligkeitsschätzung ist nicht bemerkbar. Es verteilen sich auf 45 bei Mondschein angestellte Beobach-

tungen die Abweichungen wie folgt: 19 Plus-, 15 Minuszeichen, 11 Nullwerte.

TABELLEN II*a* und II*b*. Farbenschätzungen.

Zeitraum	*n*	Farbe	Grösse	*n*	Farbe
2424231—2425977	6	$5^{c}.00$	$9^{m}.02$	6	$4^{c}.33$
2426001—2428169	6	4.42	9.44	6	5.08
	12			12	

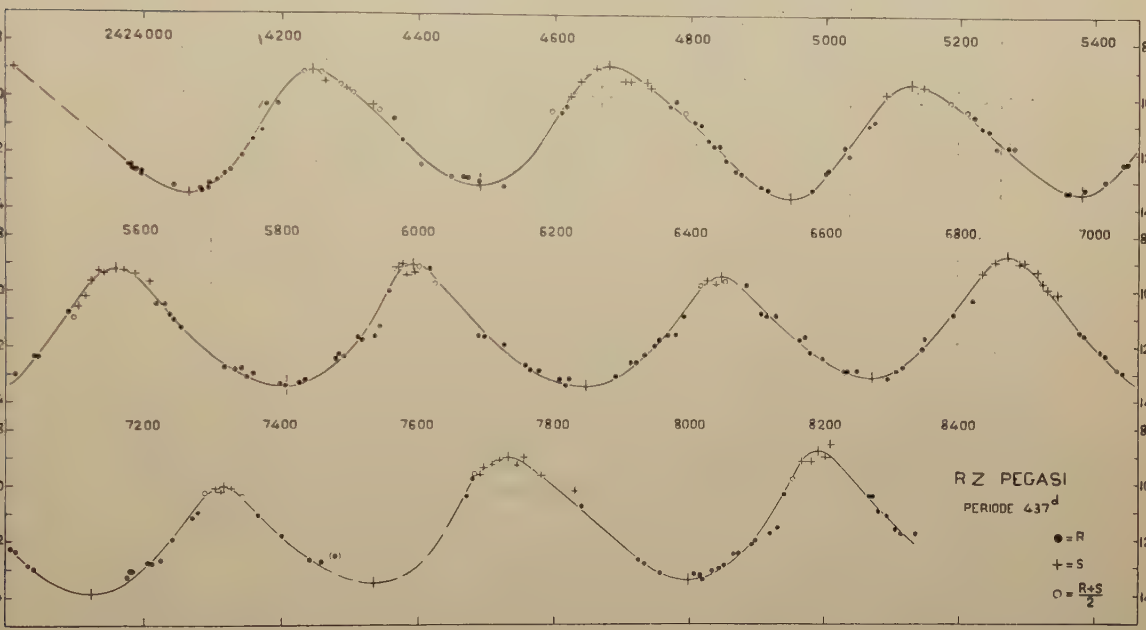


Fig. 1.

Die Tabelle III enthält die aus der Kurve abgelesenen Epochen der Minima *m* und der Maxima *M*. Die Spalte *R* wurde mit den einfachen Elementen:

$$2426434 + 437^{d}.1\ E \text{ (für die Maxima)}$$
$$2426246 + 437\ .1\ E \text{ (für die Minima)}$$

gerechnet.

Für $\frac{M-m}{P}$ findet man 0.430.

SCHNELLER's Katalog für 1937 gibt den Periodenwert 440^{d}.

Die extremen Werte des Lichtwechsels sind:

$$\left.\begin{array}{ll}\text{Minimum:} & 13^{m}.48 \pm 0^{m}.07 \\ \text{Maximum:} & 9\ .15 \pm 0\ .13\end{array}\right\}\text{(m.F.).}$$

TABELLE III.

E	Minima m				Maxima M			
	B	v	R	B—R	B	v	R	B—R
—5	242 4065	m 13.5	4060	+ 5	4244	m 9.0	4248	— 4
—4	4490	13.2	4498	— 8	4680	8.8	4686	— 6
—3	4947	13.6	4935	+12	5126	9.5	5123	+ 3
—2	5380	13.5	5372	+ 8	5558	9.2	5560	— 2
—1	5806	13.4	5809	— 3	5990	9.0	5997	— 7
0	6245	13.5	6246	— 1	6445	9.5	6434	+11
+1	6670	13.2	6683	—13	6870	8.7	6871	— 1
+2	7124	13.9	7120	+ 4	7316	10.0	7308	+ 8
+3	[7534]	[13.5]	—	—	7730	9.0	7745	—15
+4	7997	13.5	7994	+ 3	8190	8.8	8182	+ 8
		13.48				9.15		

Die Amplitude beträgt somit $4^m.33$.

Es wurde wieder der mittlere Verlauf der Lichtkurve in der Umgebung der beiden Hauptphasen durch Ablesung der Helligkeit für je 10^d abgeleitet. Die beiden Teilkurven schliessen sich, wie aus der Figur 2 ersichtlich, vorzüglich an einander an, und liefern zusammen eine glatt verlau-

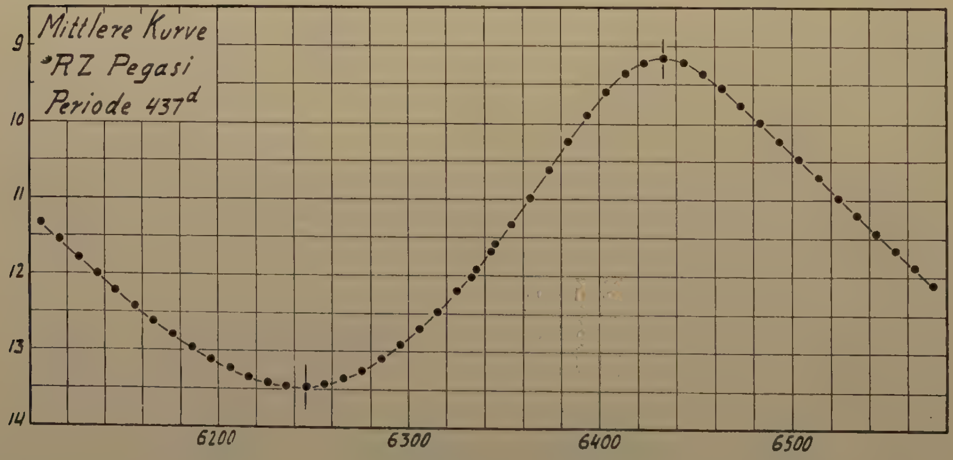


Fig. 2.

fende mittlere Kurve (Tabelle IV). Das sehr unsichere Minimum 2427534 musste unberücksichtigt bleiben.

TABELLE IV. Die mittlere Kurve.

Phase	v	Phase	v	Phase	v	Phase	v	Phase	v
d	m	d	m	d	m	d	m	d	m
−140	11.33	− 40	13.23	+ 60	12.70	+160	9.55	+260	10.53
−130	11.55	− 30	13.34	+ 70	12.47	+170	9.31	+270	10.76
−120	11.79	− 20	13.41	+ 80	12.20	+180	9.19	+280	11.00
−110	11.99	− 10	13.46	+ 90	11.92	+190	9.16	+290	11.24
−100	12.22	0	13.48	+100	11.59	+200	9.23	+300	11.47
− 90	12.42	+ 10	13.43	+110	11.27	+210	9.39	+310	11.70
− 80	12.62	+ 20	13.36	+120	10.90	+220	9.59	+320	11.94
− 70	12.79	+ 30	13.26	+130	10.55	+230	9.82	+330	12.16
− 60	12.96	+ 40	13.10	+140	10.17	+240	10.05		
− 50	13.11	+ 50	12.91	+150	9.83	+250	10.29		

Die Streuung in der Nähe von 90^d erreicht die Werte:

	m	M	Mittel
im aufsteigenden Aste:	$0^m.250$	$0^m.239$	$0^m.244$
im absteigenden Aste :	0 .124	0 .262	0 .193
Mittel:	0 .187	0 .250	

Das Verhältnis der Streuungen $0^m.244$ und $0^m.193$ ist 1.26, das Verhältnis der durchschnittlichen Geschwindigkeiten des Lichtwechsels bei Auf- und Abstieg ist 1.32.

Utrecht, Mai 1938.

Anatomy. — *On Peaks occurring in Frequency Curves of the Cephalic Index and their supposed Significance in indicating the Component Races and Subracial Groups underlying the Population.* I. *Exposition of the Problem. General Remarks.* By W. A. MIJSBERG (Batavia).

(Communicated at the meeting of May 28, 1938).

Since 1930 ARIËNS KAPPERS in the Proceedings of the Koninklijke Akademie van Wetenschappen te Amsterdam has published a series of papers on cephalic index frequency-curves of a great many populations and the conclusions to be drawn from the form of these curves with regard to the races and subracial groups which have taken part in constituting each population. He started this series with an investigation on the dimensions of the heads of 141 Armenians measured by him [1]). This material consisted partly of students of the American University of Beirut, partly of people in the towns of Beirut and Constantinople. The headform of 5 of the students was separated from the rest, as these students came from a part of the country where Persian admixture was probable. The other individuals, 97 males and 39 females, were studied together. The cephalic index of each member of this group was computed, classes comprising one index-unit each were formed and the number of individuals belonging to each class, i.e. the class-frequencies, were represented graphically. These frequencies are as follows:

Index-classes	Frequencies	Index-classes	Frequencies
75.0—75.9	1	85.0—85.9	12
76.0—76.9	1	86.0—86.9	17
77.0—77.9	4	87.0—87.9	12
78.0—78.9	0	88.0—88.9	9
79.0—79.9	1	89.0—89.9	7
80.0—80.9	3	90.0—90.9	6
81.0—81.9	5	91.0—91.9	5
82.0—82.9	11	92.0—92.9	3
83.0—83.9	19	93.0—93.9	6
84.0—84.9	13	97.0—97.9	1

From these figures it appears that the highest frequencies are in index-classes 83.0—83.9 and 86.0—86.9 respectively. As the two classes lying between contain a smaller number of individuals the frequency-curve

[1]) C. U. ARIËNS KAPPERS, Contributions to the Anthropology of the Near-East. I. The Armenians. Proc. Kon. Akad. v. Wetensch., Amsterdam, **33** (1930).

presents two peaks corresponding with the former classes (see Fig. 1 and 2). Additional peaks correspond with index-classes 77.0—77.9 and 93.0—93.9. Consequently the frequency-curve is somewhat irregular, it is not as smooth as an ideal curve of variation should be.

Now the anthropologist is accustomed to irregularities of the frequency-curves constructed by him. He is well aware of the fact that the material studied by him is as a rule far too limited to yield results corresponding with the results of random sampling. In the case of KAPPERS' Armenians deviations from the ideal frequency-curve may be expected almost with certainty for the following reasons.

In the first place his material cannot be regarded as representing a homogenous group, as his Armenian students probably came from many different parts of the country as did undoubtedly the Armenian population of the big town of Contantinople.

Secondly in his material both sexes are represented; KAPPERS studied them together since the average length-width indices differed only very little. Now it is a well-known fact that as a rule sexual differences can be found in the cephalic index of all populations of which enough individuals have been investigated.

In the third place KAPPERS studied the distribution of 136 observations over 20 index-classes. Without knowing which should be the relation between the total number of observations and the number of classes in order to get a good notion of the form of the ideal distribution (YULE [1]), p. 87, mentions a number of one thousand observations at least), one may expect almost with certainty that in distributing 136 observations over 20 classes severe deviations from the ideal curve will be brought about.

For these reasons one is inclined to ascribe the occurrence of peaks in the frequency-curve of the cephalic index of KAPPERS' Armenians to different outer circumstances; one hesitates to accept KAPPERS' view, according to which these peaks should be considered as indicating the presence of different subracial groups among the Armenians.

In his first paper KAPPERS did not draw the latter conclusion, but in a second paper on this subject [2]) he did. It was the study of the cephalic index of other populations in Western-Asia and Lower-Egypt which induced him to do so. The frequency-curve of the cephalic index in 206 Lebanese presented namely two peaks at about the same places as in the Armenians, but moreover an additional peak at 80.0—80.9 was present. This occurrence of a larger number of individuals with lower cephalic indices accounts for the fact that the mean index in the Lebanese was found somewhat lower than in the Armenians. This difference, in the

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1) G. U. YULE, An Introduction to the Theory of Statistics. 10th Ed. London (1932).

2) C. U. ARIËNS KAPPERS, Contributions to the Anthropology of the Near-East. II. The Spread of brachycephalic races. Proc. Kon. Akad. v. Wetensch., Amsterdam, **33**, N°. 8 (1930).

following quotation indicated as "the first factor", KAPPERS explained as follows (l.c., p. 804): "The first factor is due to the fact that the brachycephalic population of the Lebanon is mixed with a subbrachycephalic (ind. 80—80.9) population as appears from a superposition of the Armenian and Lebanese (male) frequency curves. The question as to the origin of this admixture is answered by the second curve that gives the frequency of the length-width index as found at the foot of the mountainous country on the desert border from Homs to Hauran, including the population of Damascus, a large town, the population of which is rather mixed but shows a typical top at 80—80.9. Superposing the Lebanese and desert border curve we see that the main top of the latter coincides exactly with the additional top in the Lebanese curve, so that apparently the population of Damascus causes the extra top of the Lebanese population."

The last sentence of the foregoing quotation is the first place where KAPPERS gives expression to his view that peaks occurring in frequency-curves of the cephalic index correspond with the main tops in the curves of the races or subracial groups which have taken part in constituting the population in question, or at least point to the presence of such races, for, as appears from later papers, the exact place of the peak may vary a little. Now the sentence in question does not give any proof whatever of the exactness of this fundamental surmise. And it was in vain that in later papers I sought for arguments. In the first chapter of his monograph "An Introduction to the Anthropology of the Near East" [1]) KAPPERS with regard to the frequency-curves of the cephalic index writes (l.c., p. 6): "Whereas averages always more or less take away the differences existing in a group, or only express them by their spread and deviations, a frequency curve gives the complete analysis of its group. This is the more important as every group — even such as are stamped with a racial name — is always mixed. In these groups, provided they are large enough (my experience is that only groups of a hundred or more individuals give reliable relations), we may distinguish the chief component or highest peak from its variations or admixtures. Especially the admixtures may vary a great deal, according to the country where the group lives. While with racial groups [2]) usually one and the same peak is found, it is a curious fact that in some groups — e.g. the Armenians and Khaldeans (see below) — the are always two high peaks, which have a certain constancy."

This moreover not only holds good in studying different sets of individuals representing the same population, but also in comparing different more or less closely related groups. KAPPERS [3]) even found the same peaks

[1]) Noord-Hollandsche Uitgeversmaatschappij, Amsterdam (1934).

[2]) This is in opposition to the preceding thesis according to which every group is mixed.

[3]) C. U. ARIËNS KAPPERS, The spread of primitive humanity and its links with the more differentiated races, as revealed by cephalic and cranial index curves. Proc. Kon. Akad. v. Wetensch., Amsterdam, **39**, N^{0}. 10 (1936).

in groups which at present are geographically separated as far as the Melanesians on the one hand and the African negroes on the other. The peaks present in the cephalic index frequency-curves of these groups confirm their racial relationship, established by modern anthropological research (VON EICKSTEDT [1])).

Such are the facts on which KAPPERS' explanation of the occurrence of peaks is founded. The existence of these facts cannot be denied. But still one may ask whether the peaks observed in the frequency-curve of a population are a typical character of this population, or if they are brought about by chance only or by some other outer cause. The circumstance emphasized by KAPPERS, that the peaks have a certain constancy, may plead in favour of the first possibility; still it seems worth while to study their reliability from a statistical point of view. In case the peaks should prove reliable, the difficulty as to their explanation again arises. For KAPPERS' explanation certainly is a possibility; but he has given no proof whatever of its correctness. Therefore it seems necessary to consider other possible explanations too and only in case no other effort succeeds I should be willing to admit the exactness of KAPPERS' surmise. For one should not lose sight of the circumstance that in anthropological research different groups cannot be sufficiently differentiated by one criterion only. It is generally accepted in anthropology that races are characterized by the presence of a complex of characters, each of which may separately occur in other races too; in the given combination they are however typical of the race in question. Consequently it is very remarkable that in KAPPERS' anthropological work not only one character — important though it may be — should suffice, but that even the distribution of this one character over the index-classes of the group should enable one to tell its racial components!

In the case of his 136 Armenians KAPPERS even has gone so far as to try and separate the frequency-curve into two composing curves each showing only one top which corresponds with one of the main peaks of the original curve. He has not drawn the component curves themselves but he has given their statistical dates. With regard to the main tops of the original curve he states namely [2]): "The probability that these tops have the value of a reality has been tested by a mathematician and confirmed so that there is some reason to accept that among the Armenians two essentially different groups occur, one with an average l.w. index of 83.48 ($\pm$ 1.16), the other with an average l.w. index of 86.53 ($\pm$ 1.49)." He has repeated this statement in his "Introduction to the Anthropology of the Near East" in writing (l.c., p. 14): "The probability calculation of my own Armenians (men and women) shows that there are two groups, one of 83.48 ($\pm$ 1.16) and one of 86.53 ($\pm$ 1.49)".

[1]) E. Freih. VON EICKSTEDT, Rassenkunde und Rassengeschichte der Menschheit. Stuttgart (1934).

[2]) Proc. Kon. Akad. v. Wetensch., Amsterdam, **33**, 808 (1930).

As I wanted to investigate if in combining these two curves a compound curve provided with two peaks should result, I tried to reconstruct the ideal curves which are characterized by the above mentioned dates. Now however the difficulty arose, that KAPPERS has not mentioned which measure has been put behind the $\pm$ sign. One is inclined to suppose that it is either the standard error of the mean or its probable error. In the first case the standard deviation of each series should be $\sqrt{n}$ times as large. Now supposing each group to consist of about an equal number of individuals, $\sqrt{n}$ attains the value 8. Consequently the standard deviation of the first group amounts to about 9, that of the second to 12. Taking $M \pm 3\sigma$ as the limits between which practically all values are to be found, the first group should range from index-class 56 to 110, the second from 50 to 122. In case the probable error of the mean should have been given, these ranges should still be wider. From this rough calculation it is evident, that the measure of deviation given by KAPPERS cannot be a measure of the mean; it must be the standard deviation or the probable error of each series.

Supposing the distribution of frequencies in each component group to be an ideal one, it is possible, with the help of the standard error and of the probable error respectively, to calculate which area of each curve falls on each class. Therefore it is possible to express each class-frequency as a fraction of the total number of individuals of the group. I made use of JOHANNSEN's table ([1]), p. 77), but other tables will do as well. As a matter of fact the two curves overlap. In the compound curve, bearing on the combined groups, the frequencies of the classes 83.0—83.9 and 86.0—86.9 must be the same as in the original curve of the 136 Armenians, investigated by KAPPERS, as in these classes the main tops of the latter are situated. These frequencies are 19 and 17 respectively. Therefore the sum of the fractions of the number of individuals which in each of the two component curves fall in index-class 83.0—83.9 must be 19, whereas in class 86.0—86.9 this sum must be 17. From these equations the number of individuals of each component group can be computed. If in this calculation it is assumed that KAPPERS has given the probable error of the groups, the number of individuals is 59 and 78 respectively, or 137 in total. If KAPPERS had given the standard deviations of the groups a rough calculation shows that the numbers should be about 51 and 61, in total 112. As in the first case the total number of individuals agrees very well with the number of Armenians measured by KAPPERS (136), whereas in the second it does not agree, we may assume that the measure of deviation given by KAPPERS is the probable error of each series. Taking this for granted I have calculated the frequencies of each class in the theoretical curves. The results are as follows:

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[1]) W. JOHANNSEN, Elemente der exakten Erblichkeitslehre, 3. Aufl. Jena (1926).

79.0—79.9	80—80.9	81—81.9	82—82.9	83—83.9	84—84.9	85—85.9	86—86.9	87—87.9	88—88.9	89—89.9	90—90.9	91—91.9	
1	3	7	11	13	11	7	3	1	—	—	—	—	($n =$ 57)
—	—	1	3	6	9	13	14	13	9	6	3	1	($n =$ 78)
1	3	8	14	19	20	20	17	14	9	6	3	1	($n =$ 135)

From these figures it appears that the usual way of rounding off fractions has decreased the number of individuals of the first group from 59 to 57. Still the total number (135) corresponds very well with the number (136) in KAPPERS' original curve.

In fig. 1 the two theoretical component curves are represented by a

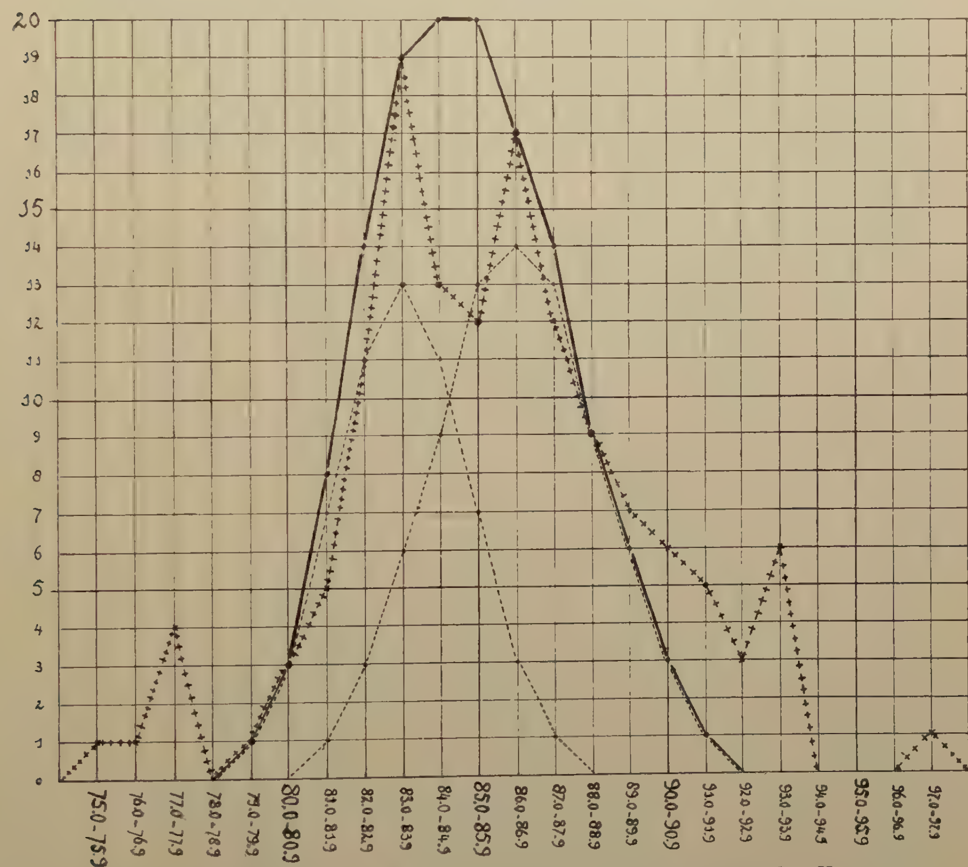

Fig. 1. Frequency-curve of the Cephalic Index of 136 Armenians after KAPPERS (crossed line). Ideal frequency-curves of the two groups which, according to KAPPERS' dates, should occur in these Armenians (dotted lines). Frequency-curve resulting from combining the latter two theoretical groups (full line).

dotted line. The curve resulting from combining both groups is drawn with a full line. Moreover the curve of the 136 Armenians measured by KAPPERS is figured with a crossed line. It appears from this figure that the frequencies of the 83.0—83.9 and 86.0—86.9 classes in the original

curve and in the resulting theoretical curve are the same. But the depression which in the original curve lies in between, is not present in the theoretical one. In stead of it the latter shows a flattened top in the corresponding area.

I do not know in which way KAPPERS has succeeded in separating his material into two groups. In my opinion such an enterprise seems wholly arbitrary, if at least he has been guided by the cephalic index only. At any rate the two groups separated by him and characterized by the means 83.48 ± 1.16 and 86.63 ± 1.49 cannot have shown a nearly normal distribution.

It may be added that also in the case it is the standard deviation which KAPPERS has put behind the $\pm$ sign, in the resulting theoretical curve no depression appears between the frequencies of the classes 83 and 86.

From the foregoing I must conclude that the probability calculation does not support KAPPERS' view according to which the presence of two main peaks should prove that the Armenians consist of a mixture of two different groups.

In the beginning of this paper doubt has already been expressed with regard to the question if the relation between the number of classes and the total number of Armenians investigated by KAPPERS is such as to justify the expectation that the frequency-curve based on this material will give a good idea of the normal distribution in the whole of the population. According to YULE (l.c., p. 79), the conditions which guide the choice of the magnitude of the class-interval "will generally be fulfilled if the interval be so chosen that the whole number of classes lies between 15 and 25". Now in the case of KAPPERS' Armenians the number of classes is 20. But l.c., p. 87, YULE states: "When, in any actual case, the number of observations is considerable — say a thousand at least — the run of the class-frequencies is generally sufficiently smooth to give a good notion of the form of the ideal distribution; with small numbers the frequencies may present all kinds of irregularities, which, most probably, have very little significance". As in the case of KAPPERS' Armenians this number is only 136, the conclusion is obvious.

More reliable results might possibly be attained in increasing the frequencies by enlarging the class-intervals. Therefore I took together every two successive classes of KAPPERS' curve. This of course may be done in two different ways. The results are shown on the following page.

It appears from these figures that the curves which could be drawn should be smoother than the original one; the curve corresponding with the first row of figures would even present one top only. Now it is self-evident that in enlarging the class-intervals some of the irregularities of the first curve must necessarily disappear. For in drawing the new curve we treat all the values assigned to each of the new classes as if they were equal to the mid-value of the new class-interval. Therefore if

Index-classes	Frequencies	Index-classes	Frequencies
75.0–76.9	2	74 0–75.9	1
77.0–78.9	4	76.0–77.9	5
79.0–80.9	4	78.0–79.9	1
81.0–82.9	16	80.0–81.9	8
83.0–84.9	32	82.0–83.9	30
85.0–86.9	29	84.0–85.9	25
87.0–88.9	21	86.0–87.9	29
89.0–90.9	13	88.0–89.9	16
91.0–92.9	8	90.0–91.9	11
93.0–94.9	6	92.0–93.9	9
95.0–96.9	—	94.0–95.9	—
97.0–98.9	1	96.0–97.9	1

an enlarged class corresponds with two successive classes of the original curve in which a peak and a valley respectively fall, it is clear that this contrast will no longer be visible in the new diagram. Hence the disappearance of many of the original irregularities does not prove that the latter were not trustworthy. The reliability of these peaks must be studied in another way which will be discussed below.

First however I will consider the way in which Miss KEERS [1]) has tried to diminish the irregularities of the curve without enlarging the class-intervals. This author assigns as class-frequency the mean of the observed frequencies of the class in question, the preceding one and the following one. In the Armenians studied by KAPPERS the class-frequencies according to this procedure would be as follows:

Index-classes	Frequencies (calculated after KEERS)	Index-classes	Frequencies
75.0–75.9	1	85.0–85.9	14
76.0–76.9	2	86.0–86.9	14
77.0–77.9	2	87.0–87.9	13
78.0–78.9	2	88.0–88.9	9
79.0–79.9	1	89.0–89.9	7
80.0–80.9	3	90.0–90.9	6
81.0–81.9	6	91.0–91.9	5
82.0–82.9	12	92.0–92.9	5
83.0–83.9	14	93.0–93.9	3
84.0–84 9	15	94.0–94.9	2

The frequency-curve which might be drawn from these figures would present hardly any irregularity. Especially all deflexions in the middle

[1]) W. KEERS, Een anthropologisch onderzoek van de Karo-Batak. Geneeskund. Tijdschr. v. Nederl.-Indië, Vol. 77, 328 (1937).

part of the curve have disappeared, on the other hand its top is very flat. The way in which these changes have been brought about can be explained in the easiest manner by taking the calculation of the frequencies of the classes 83.0—83.9 and 84.0—84.9 as an example. The frequency of class 83 according to this method is one third of the sum of the numbers of individuals which according to observation should belong to the classes 82, 83 and 84 respectively. In the same way the frequency of class 84 is one third of the sum of the observed frequencies of the classes 83, 84 and 85. Therefore the difference between the frequencies of classes 83 and 84 according to this method is one third of the difference between the numbers of individuals which according to observation should belong to the classes 82 and 85. The difference between the observed frequencies of classes 83 and 84 however does not play any part in constituting the difference between the frequencies of the same classes computed after KEERS!

The latter fact is in my opinion of great importance. I for my part cannot approve of a method which effaces any difference between the observed frequencies of two successive classes, without even putting the question if this observed difference be reliable. However great the observed difference may be, it can never appear from the frequencies of the successive classes in question computed after KEERS. The latter author applied this method in order to reduce the influence of errors of observation. As has been explained before, it does more; I think it overshoots the mark!

In the foregoing some methods have been discussed to reduce irregularities of the frequency-curve which might have occurred as a fluctuation of sampling. They were more or less unsuccesful. It goes without saying that the best way in which this object could be attained consists in increasing the number of observations. Now KAPPERS has given the frequency-curves of the cephalic index of 530 male Armenians and of 326 female Armenians, measured by himself, CHANTRE, KRISCHNER and VON ERCKERT (in Introduction to the Anthropology of the Near East, l.c., p. 15). In Fig. 2 I have reproduced these curves, slightly altered, and I have added the curve bearing on the 136 male and female Armenians of his first paper.

It is interesting to note that the curve of the 530 males is the most regular one of the three. The two main tops are still recognizable but the left one has shifted one class-interval to the right, while the valley lying between is far less pronounced.

Although we should bear in mind that the three curves of Fig. 2 are not independent ones, as the individuals represented in the curve of the 136 Armenians, are also included in the curves of the male and the female Armenians respectively, still one gets the impression that in increasing the number of observations an increasing tendency towards fusion of the two main peaks can be detected. Therefore the question arises if perhaps by further increase of the number of observations a curve with one top

only should have come into existence. This question cannot be answered, but it is possible to study the odds against the occurrence of two main tops in an other investigation of the cephalic index of the Armenians.

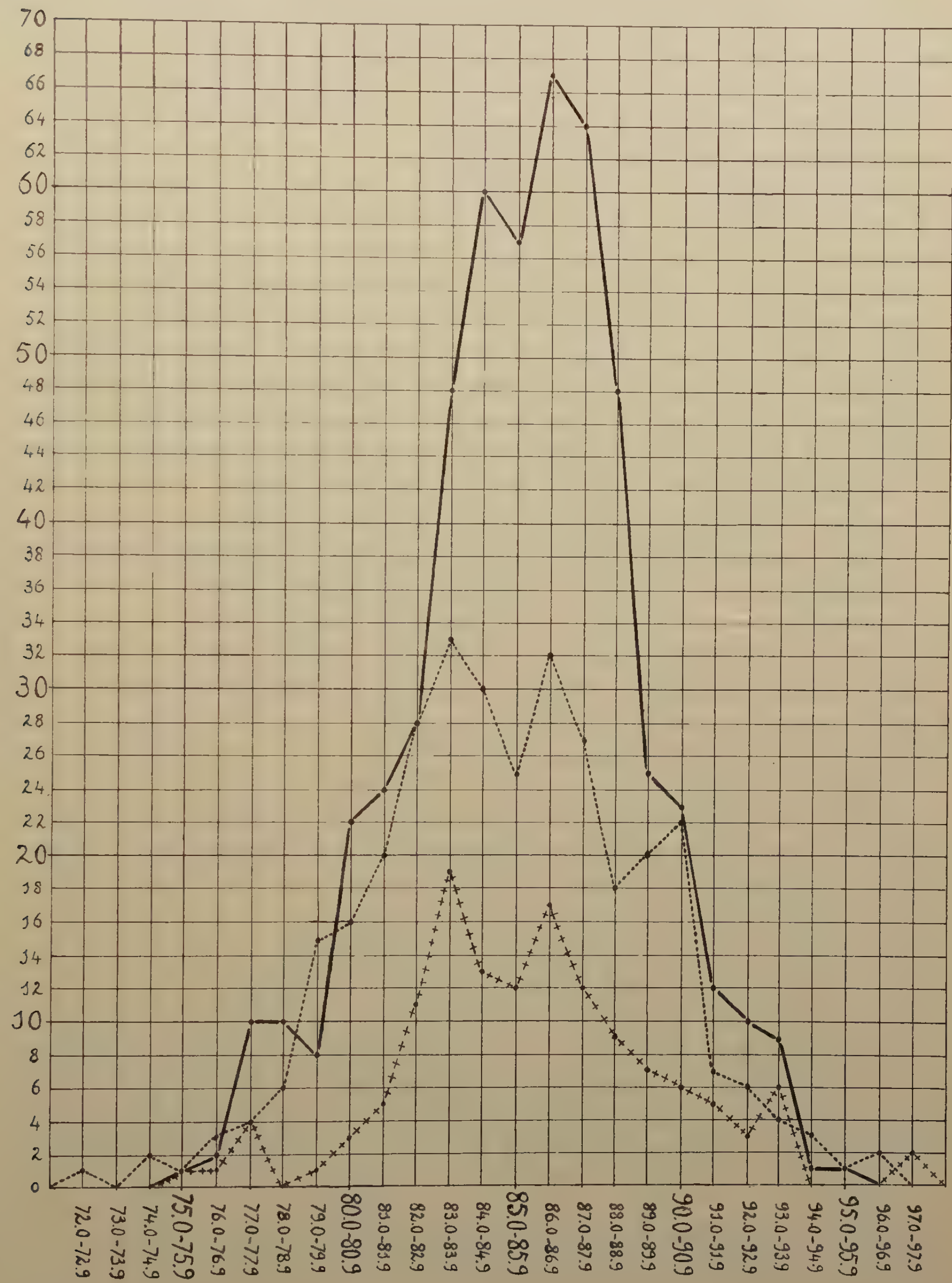

Fig. 2. Frequency-curves of the Cephalic Index of 530 male Armenians (full line), of 326 female Armenians (dotted line) and of 136 Armenians of both sexes (crossed line), after KAPPERS.

Let us first consider the curve of 136 individuals of both sexes. Here, as in all three curves, the deepest point of the valley between the two main peaks corresponds with class 85.0—85.9. The frequency of this class is 12 (N_1), the number of individuals not belonging to this class (N_2) is 124. The total number of observations being N, the standard deviation of the frequency of class 85 is $\sigma_{85} = \sqrt{\frac{N_1 \times N_2}{N}} = \sqrt{\frac{12 \times 124}{136}} = 3.31$ [1]. In the same way $\sigma_{83} = 4.04$; $\sigma_{84} = 3.43$; $\sigma_{86} = 3.86$.

From these figures it appears that in a following investigation of 136 Armenians of the same sexual proportion the frequency of class 85 will almost with certainty be found lying between the limits $12 \pm 3\,\sigma_{85}$, i.e. between the values 2 and 22. We may therefore conclude that the valley in the frequency-curve of KAPPERS' 136 Armenians which corresponds with class 85.0—85.9 may be explained as a fluctuation of sampling.

In the case of the 530 male Armenians the frequencies of the classes 84, 85 and 86 with their respective standard deviations are 60 ± 7.29; 57 ± 7.13; 67 ± 7.65. Again it seems quite possible that the two main peaks and the valley lying between may be explained as fluctuations of sampling.

In the foregoing way it is however impossible to study the probability of the absence of two peaks in a following investigation. But this may be done as follows.

In the series of 136 Armenians the average cephalic index according to KAPPERS' data must fall in index-class 85.0—85.9. Therefore the frequencies of classes 83 and 86 are higher, or those of classes 84 and 85

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[1]) One might also have used the equation $\sigma = \sqrt{pqN}$ in which p is the fraction of the total number of individuals belonging to class 85, hence $p = \frac{N_1}{N}$, and q is the fraction not belonging to it, hence $q = \frac{N_2}{N}$.

$$\sigma = \sqrt{pqN} = \sqrt{\frac{N_1}{N} \times \frac{N_2}{N} \times N} = \sqrt{\frac{N_1\,N_2}{N}}.$$

Also the well-known equation might have been used $\sigma_1 = \sqrt{\frac{P_1\,(100 - P_1)}{N}}$ in which σ_1 is σ in percentages of N, hence $\sigma_1 = \frac{\sigma}{N} \times 100$, and P_1 the class-frequency in percentages of N, hence $P_1 = \frac{N_1}{N} \times 100$.

$$\sigma_1 = \sqrt{\frac{P_1\,(100 - P_1)}{N}} = \sqrt{\frac{\frac{N_1}{N} \times 100 \left(100 - \frac{N_1}{N} \times 100\right)}{N}} =$$

$$= \frac{100}{N} \sqrt{\frac{N_1\,(N - N_1)}{N}} = \frac{100}{N} \sqrt{\frac{N_1\,N_2}{N}} = \frac{\sigma}{N} \times 100.$$

are lower than ought to be the case in normal distribution. The difference between the observed frequencies of classes 86 and 85 is 5. What is the standard-deviation of this difference? According to YULE (l.c., p. 210) the standard-deviation of the difference of two variables is

$$\sigma_{Diff.} = \sqrt{(\sigma_1^2 + \sigma_2^2 - 2r\sigma_1\sigma_2)},$$

in which σ_1 and σ_2 are the standard-deviations of the variables in question, r their correlation-coefficient. σ_1 and σ_2 have already been calculated, r is as yet unknown. It is however clear that r must have a negative value. For if the total number of observations is constant and the frequency of class 85 increases, the number of individuals not belonging to this class must necessarily decrease. Hence also the frequency of class 86 will tend to decrease. If the latter tendency be considerable or not might only be derived from the study of a series of investigations of Armenians, each extending over the same number of individuals. As such series are not available, we must confine ourselves to stating that between the frequencies of neighbouring index-classes a negative correlation probably exists, the exact value of which is unknown. Consequently we may assume that $\sigma_{Diff.} > \sqrt{(\sigma_1^2 + \sigma_2^2)}$. In the case of the class-frequencies 86 and 85 $\sigma_{Diff.} > \sqrt{3.86^2 + 3.31^2}$, i.e. $\sigma_{Diff.} > 5.06$. The difference of the frequencies ($D = 5$) is therefore less than ·99 times its standard-deviation.

If in a following investigation a difference between the frequencies of classes 86 and 85 should be found which equals or is smaller than D — ·99 × 5.06, the class-frequency 86 should not correspond to a peak in the curve, as D should be 0 or even attain a negative value. If therefore the difference should equal or be smaller than $D - \cdot 99\sigma_{Diff.}$ the frequency of class 86 should already be smaller than that of class 85. Now the probability of the latter occurrence can be calculated from a normal curve of variation the mean of which is D and the standard-deviation of which is $\sigma_{Diff.}$. From the tables of normal distribution (e.g. JOHANNSEN, l.c., p. 77) the probability that in a following observation a difference will be found which equals or is smaller than $D - \cdot 99\ \sigma_{Diff.}$ appears to be 16.12 %. The odds against such an occurrence are about 5 to 1. Therefore the odds against reliability of the peak corresponding with index-class 86.0 — 86.9 are about 1 to 5.

In the same way the difference between the frequencies of index-classes 83 and 85 is $D \pm \sigma_{Diff.}$, or $7 \pm > 5.22$. The difference here is less than 1,34 times its standard-deviation. The probability that in a following investigation of 136 Armenians of the same sexual proportion the difference between these class-frequencies will be found to have a value $= D - 1.34\ \sigma_{Diff.}$ or even smaller than that, is about 9 %. The odds against reliability of the peak at index-class 83.0 — 83.9 consequently are about 1 to 10.

From these probabilities it will be clear that I do not agree with KAPPERS' statement already cited: "The probability that these tops have the

value of a reality has been tested by a mathematician and confirmed......". Usually in statistics far greater probabilities are required before one is inclined to assume certainty.

In the case of the 530 male Armenians of Fig. 2 the difference between the frequencies of classes 84 and 85 is $3 \pm > 10.20$. Hence this difference is less than ·29 times its standard-deviation. The odds against reliability of the peak at class 84 consequently are about 1 to $1\frac{1}{2}$. In comparing the frequencies of classes 86 and 85 the difference appears to be $10 \pm > 10.46$; the difference is less than ·96 times its standard-deviation. The odds against reliability of the peak found at index-class 86 therefore are about 1 to 5.

From these probabilities one is compelled to conclude that it is far from certain that the two main tops occurring in all three Armenian cephalic index frequency-curves of fig. 2 are typical of this population. It is especially remarkable that in increasing the number of observations the odds against reliability of these peaks appear to increase too. Therefore there is a not unconsiderable possibility that the real frequency-curve of the cephalic-index in the Armenians should not show two tops and that their presence in the groups investigated must be ascribed to fluctuations of sampling, if not to accessory circumstances which favour an increase respectively a decrease of the frequencies of particular index-classes. The latter possibility will form one of the subjects of a following paper.

In later papers KAPPERS has called the peaks occurring at index-classes 83 and 86 "associated index peaks", because both of them, together or separately, characterize his Central-Asiatic race [1]). Therefore the 83 and 86 peaks should each represent a subracial group.

In frequency-curves of other populations peaks may occur which according to KAPPERS should be ascribed to different races. In case such peaks do not lie too far from each other their reliability can be investigated in the manner described here; if their distance is larger another way must be followed which will be discussed in a following paper .

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[1]) Proc. Kon. Akad. v. Wetensch., Amsterdam, **37**, 612 (1934); **38**, 686 (1935).

Physics. — *Determination of the rate of infection in tuberculosis.* By Dr. G. C. E. BURGER, head of the Health Service of the N.V. Philips at Eindhoven, and Dr. H. C. BURGER, Lecturer in physics at the University of Utrecht.

(Communicated at the meeting of May 28, 1938.)

The tuberculin reaction according to PIRQUET and its later modifications (including tuberculin ointment reaction according to MORO, HAMBURGER) have not only been found to be of great significance for the diagnosis of tubercular affections, but have also made it possible to obtain with the aid of statistics an insight into the spreading of tuberculosis among the different classes of the population. The quantitative utilization of these statistics of tuberculin reaction has practically always been confined to comparing the percentage of the positive (and negative) tuberculin reactions for different classes of the population.

Recently an important attempt has been made by MUENCH in America and by STRAUB and particularly by HEIJNSIUS VAN DEN BERG in Holland to obtain from the data regarding the sensitivity to tuberculin, with the aid of mathematics, some information regarding the rate of infection by tuberculosis [1]).

If at a certain moment a group of persons of different ages is examined by means of a tuberculin test and a graph is made of the percentage of positive reactions as a function of the age, great regularity will be noticed here. This phenomenon caused HEIJNSIUS VAN DEN BERG to arrive at the conclusion that there existed a constant "factor". He assumed that this constant factor was formed by a constant rate of infection during a number of years of life. On the basis of a uninfected persons, and assuming that annually x % of these individuals becomes infected, there are then after 1 year:

$$a - \frac{x}{100}\,a \text{ or } a\left(1 - \frac{x}{100}\right),$$

after 2 years:

$$a\left(1 - \frac{x}{100}\right)^2$$

[1]) H. MUENCH, Journal of the American Statistical Association, Vol. XXIX, p. 25 (1934). — STRAUB, Report of the Genootsch. tot bev. v. Nat., Genees. en Heelk. Ned. Tijdschr. v. Geneesk., p. 1540 (1936). — HEIJNSIUS VAN DEN BERG, Report of meeting of the Ver. Ned. T. b. Artsen. N. T. v. Geneesk., p. 1872 (1937). — Reports of the Dutch Tuberculosis Committee (1937).

and after y years:

$$a\left(1-\frac{x}{100}\right)^{y}$$

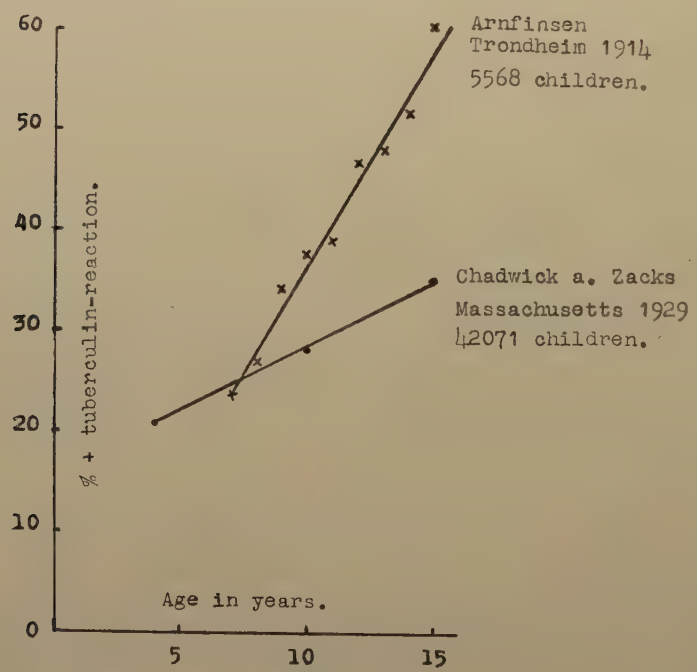
Now HEIJNSIUS VAN DEN BERG uses his formula in two different ways, viz.:

1. From a certain percentage of positive tuberculin reactions at a certain age, he calculates the annual percentage of infection, assuming that this has remained constant from birth.

2. From the comparison of the percentage of positive reactions for two age groups, he calculates the infection percentage belonging to this age group. Here again it is assumed, that the rate of infection has remained constant during the period and that the differences formed between the two ages are explained by modification of the rate of infection with the age at a certain time.

The most direct method of establishing the rate of infection would be the following. Taking as a basis a sufficiently large group of individuals

Fig. 1.



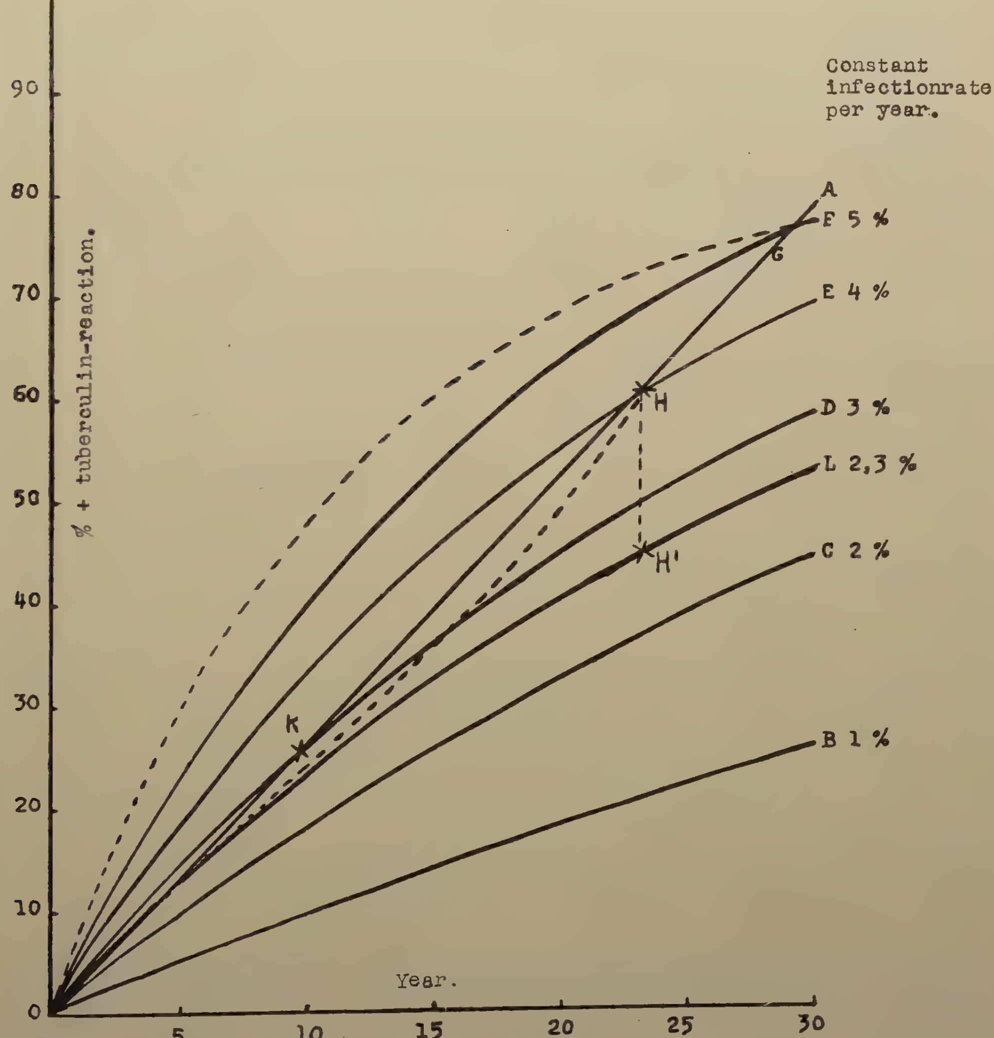
Dates taken from: E. Schröder. Ergebn. d. ges. Tbk. forschung. Bd VIII 1937.

with negative tuberculin reaction, an investigation is made regarding how many persons of this group get a reversal of their reaction in the course

of a year. As far as we know, there are no data available of sufficiently large numbers of individuals with a "normal" infection rate.

It is necessary to ask oneself whether the premise regarding the existence of a constant rate of infection is really correct. The reply must certainly be negative and this is, why we think the formula from MUENCH and of HEIJNSIUS VAN DEN BERG as incorrect. When a tuberculin curve is plotted according to the age at a certain time (year) one is struck by the regular increase with age of the number of persons with positive reaction. When shown graphically on millimetre paper the line in this case is practically linear (fig. 1). If there were a constant rate of infection in the course of years an exponential curve would be the result, as in fig. 2 (continuous lines). If the real proportions of the increase in positive tuberculin reactions

Fig. 2.



according to age at a point of time is indicated by a straight line *this signifies that in the higher age groups more positive reactions occur than must be expected according to a constant rate of infection from birth.*

Fundamentally there exist two possibilities for this:

1. At the higher ages there are more infected persons because formerly the rate of infection was greater for these individuals when they were young than it is *now* for the young people; the cause therefore lies far back in the past.

2. At older ages the rate of infection is greater at a certain point of time than at younger ages; the cause is therefore to be found in more recent events.

The first possibility is shown graphically in fig. 2, where OA shows a trend, actually found, of the number of positively reacting persons according to age at a certain point of time (increase of positive reaction by 2.6 % per year), OB a constant risk of infection of 1 % per year, OC, OD, OE and OF respectively 2, 3, 4 and 5 % per year. Point G may then be imagined as resulting from an infection risk of 5 % per year from the year of birth, point H of 4 % and point K of 3 %. In reality the proportions are different because also for the age group of $\pm$ 29 years (respectively $\pm$ 23 years and 10 years) to which points G, H and K refer, the fact applies that the risk of infection was higher during the first years after birth than later. This point K can therefore be imagined as being attained by the broken line OF, which is now no longer an exponential curve.

The second possibility can likewise be seen from the figure.

With the rate of infection of 2.3 % per year according to time and age, the point H^1 at 23 years of age would be found of the line OL; in reality H is found, because with increasing age the rate of infection has increased, resulting in a larger number of infected persons, indicated for instance at an age of 23 by the distance H^1 H, whilst one can imagine this point as being reached by the dotted line OH. In HEIJNSIUS VAN DEN BERG's considerations practically only this second possibility has been taken into account. Yet the great decrease in tuberculosis infection of the population during the last decade shows clearly, that it is just this rate of infection which is diminishing, so that a method for calculating the rate of infection must certainly take this circumstance into account.

It will be clear from this consideration that it is of fundamental significance to endeavour to find a more general solution for the calculating of the rate of infection. A surveyable and certain treatment of the by-its-very-nature statistical problem is only possible by using the tried methods of statistics, mathematical formulation being unavoidable.

The considerations following hereafter apply for a large group of persons of all ages selected at random. If we ask ourselves how many of these individuals had a certain age at a given time, for instance exactly 16 years, 0 days, 0 hours, 0 minutes, 0 seconds, their number will be

found to be practically zero. Such a case is so improbable, that we need not consider it. We must therefore consider a certain range of ages. Among the 16-year olds we could for instance understand all persons between $15\frac{1}{2}$ and $16\frac{1}{2}$. The range of one year in the age is arbitrary. If a range of one month is allowed one would have to ask after the persons aged between 16 years — $\frac{1}{2}$ month and 16 years + $\frac{1}{2}$ month.

As the range is $^1/_{12}$ of those admitted first, this number will also be $^1/_{12}$ of the first number. The first number therefore entirely characterizes the "number of 16-year olds" and is indicated by n (16).

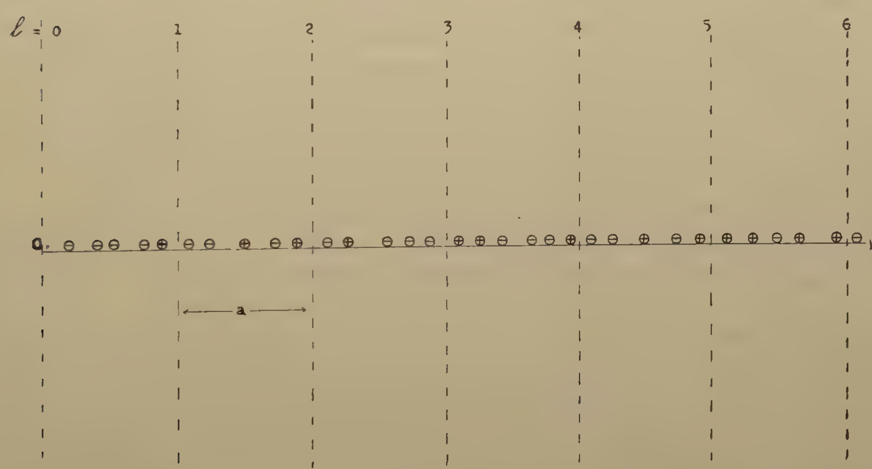
Among the $n(l)$ l-year olds there are $n_+(l)$ with a positive and $n_-(l)$ with a negative tuberculin reaction. The number of persons with positive reaction in the above fraction is therefore:

$$P(l) = n_+(l)/n(l), \text{ whilst } N(l) = n_-(l)/n(l)$$

represents the fraction with a negative reaction. It must be remembered that all the magnitudes mentioned here (n, n_+, n_-, P and N) not only depend on the *age* l, but also on the *time*, i.e. *the year*.

We must now ask ourselves what causes the magnitudes n_+ and n_- to increase or decrease in the course of time. We shall discuss these causes successively and take them into account. To make one of the most important effects clear (and it is this very effect that has been overlooked by HEIJNSIUS VAN DEN BERG) we shall use the following geometrical representation (fig. 3). At a certain time, for instance January 1st 1920,

Fig. 3.



the condition for the group of the population under consideration is shown by placing, for each individual, a point on a horizontal line at a distance from the origin 0 corresponding to the age. This point is indicated by + or —, according as the individual has a positive or negative reaction. There are just a few points in the drawing, but the statistic is only of value for many individuals, so that the actual number of points must be

very great. Such a drawing gives the above-mentioned magnitudes n_+, n_- and n as the number of + and — signs, respectively the total number of signs on a piece of the line whose length a represents the unit of age (—difference), and therefore a year for example. The representative points (+ and — signs) now all run with time to the right (for each individual the age increases with time), namely per year over the above-mentioned length a which represents one year. Besides this moving up with the time, the above mentioned important effect for the explanation of which we require fig. 3, our figure, which is the picture of the population group under consideration, shows much more. By birth individuals originate at $l=0$, i.e. the origin 0 is a source of representative signs, viz., — signs, which immediately start to move to the right with the uniform speed of all points. This occurrence of individuals with negative reaction at 0 is not of importance for our problem. Of primary importance, however, is that a — sign may change into a + sign whilst the reverse is so seldom the case that we will neglect it. Furthermore individuals, i.e. + and — signs in fig. 3, will disappear through decease. Finally + and — signs will disappear and appear through emigration and immigration.

All these effects must be taken into account and weighed up against each other in order to finally hold over what is quantitively the most important.

Through the points of fig. 3 which are generally not evenly distributed, shifting to the right, their density alters near a certain point corresponding to a certain age. If nothing else happened, the points for the age 2 ($1\frac{1}{2}$—$2\frac{1}{2}$ years) would, for instance after a year, come near the age 3 ($2\frac{1}{2}$—$3\frac{1}{2}$ years). The number of 3-year old persons with positive reaction therefore changes annually by a number corresponding to the difference between the 2- and 3-year olds and in fact increases when there are more 2-year olds then 3-year olds. If, as is usual, the annual increase of positively reacting individuals is denoted by $\partial n_+/\partial t$ and the difference in n_+ for two age groups differing one year by $\partial n_+/\partial l$ (positive when n_+ at a greater l is greater than at a smaller l) then

$$\partial n_+/\partial t = -\,\partial n_+/\partial l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We indicate by the minus sign that $\partial n_+/\partial t$ is negative (i.e. that n_+ decreases with time) when n_+ is greater for older ages than for younger ages. A relation corresponding to (1) also applies for n_-.

However, the equation (1) is incomplete. At the righthand-side there is merely the alteration of n_+ by the above-mentioned effect of "moving up the year groups". Next to this, infection must be reckoned with in the first place. As a consequence of this the number n_+ increases and n_- decreases, in both cases by the same amount because each infection increases n_+ by one and n_- decreases by one. We did not take in account the possibility, that a positive tuberculin reaction can again become negative. It seems probable that this is rather seldom, and for simplifying

the problem, we only discuss the change of a negative reaction into a positive one. The number of infections occurring per year with l-aged is proportional to their number $n\ (l)$ and is found by multiplying this number by the infection-rate $b\ (t, l)$. As also remarked by HEIJNSIUS VAN DEN BERG, this depends on the age, *but also on the time t, i.e. on the year.* Without any doubt b has considerably diminished in the course of time. The equation (1) must be completed by adding to the second term the yearly number of infections $b\, n_-$. The same number must appear in the second term of the corresponding equation for n_- with the negative sign, because, as alreary observed above, infection reduces the number n_- of non-infected persons.

Furthermore we take decease into account. This gives a reduction of n_+ as well as of n_- proportional to n_+ and n_- respectively. The proportionality factor, the death rate, is not, however, exactly the same for the positively and the negatively reacting individuals and will consequently be indicated for both cases by s_+ and s_- respectively. In the second term of the equation for n_+ and n_- the terms $- s_+ n_+$ and $- s_- n_-$ will have to be added, which represent the number of persons deceased per year.

Both s_+ and n_- depend in their turn on the age l and the time (year) t.

Finally we must also mention the effect of immigration and emigration. This althers the number n_+ per annum by an amount $\triangle_+$, the excess of settlement over departure, whilst n_- varies with $\triangle_-$. Not much more need be said about these amounts, because they are greatly dependent on local circumstances.

Resuming, therefore, the equations for n_+ and n_- read as follows:

$$\frac{\partial n_+}{\partial t} = \frac{\partial n_+}{\partial l} + b n_- - s_+ n_+ + \triangle_+ \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{\partial n_-}{\partial t} = \frac{\partial n_-}{\partial l} - b n_- - s_- n_- + \triangle_- \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Increase of number of positively or negatively reacting individuals with time	moving up of the year-groups	infection	mortality	removal

For comparison with the considerations of HEIJNSIUS VAN DEN BERG it must be observed here that he quite rightly leaves mortality and removal out of consideration. But he furthermore assumes that the state of affairs is stationary, i.e. that it does not change with time. This amounts to setting at zero the left-hand side of this equation [1]. It will be seen later that this is not allowable.

It is necessary to deduce from equations (2) and (3) applying for the

[1]) The equation (3) can therefore be solved in a simple manner and it is this solution that is used by HEIJNSIUS VAN DEN BERG.

numbers n_+ and n_- of the persons with respectively positive and negative reaction, a relation for the fraction already mentioned $P = \frac{n_+}{n}$ of all persons of the age l having a positive reaction. As a matter of fact this fraction is the magnitude that observation gives or should give us. The calculation that leads to the relation P is of no importance for understanding the problems dealt with.

The result is that we find the required rate of infection b expressed as follows in observable magnitudes:

$$\underset{\text{rate of infection}}{b} = \underset{\text{increase with age}}{\frac{1}{1-P}\frac{\partial P}{\partial l}} + \underset{\text{increase with time}}{\frac{1}{1-P}\frac{\partial P}{\partial t}} + \underset{\text{mortality}}{P(s_+ - s_-)} + \underset{\text{removal}}{P\left(\frac{\Delta_-}{n_-} - \frac{\Delta_+}{n_+}\right)}. \quad (4)$$

We shall discard the last term. As a rule nothing can be said of the "removal term" and there is therefore no reason to assume that those who have removed differ in composition from those present.

The term referring to mortality is certainly positive, because the mortality-rate of the positively reacting persons s_+ is greater than that of the negatively reacting individuals s_-.

As, however, s is of the order of magnitude of a few per thousand per annum and a small fraction of the mortality is due to tuberculosis, we find for the "morality term" in (4) a value of only a few tenths per thousand. As the other terms are of the order of a few per cent per annum, the about 100 times smaller "mortality term" can be discarded with respect to them.

Finally the equation (4) condenses to the following simple relation for the rate of infection b:

$$b = \frac{1}{1-P}\frac{\partial P}{\partial l} + \frac{1}{1-P}\frac{\partial P}{\partial t} \text{ or: } \boxed{b = \frac{1}{1-P}\left(\frac{\partial P}{\partial l} + \frac{\partial P}{\partial t}\right)} \quad . \quad (5)$$

If P, the fraction of those reacting positively, were known as a function of the age l and the year t, the problem would be solved in this way. But, unfortunately, the observation material is so scarce and at the same time so contradictory that we shall have to confine ourselves to a rough estimation of b.

Unfortunately, we shall have to consider as unattainable for the present the ideal case, namely establishment of b as a function of the age and year, for different countries and groups of population.

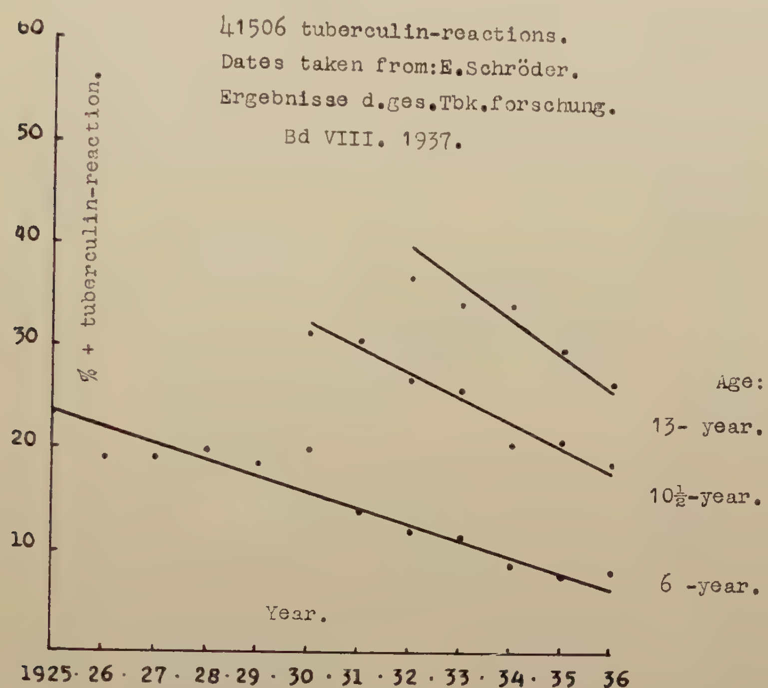
In fig. 4 we give a graphic representation of the percentage of children with positive reaction as a function of the year, according to E. SCHRÖDER. The slope of the line is greater at an older age, but in connection with the uncertainty of the data we shall confine ourselves for the present to the average life of 10 years and a time corresponding to the year 1930.

The slope then gives us for the reduction of P with the time:

$$\frac{\partial P}{\partial t} = -2.5\ {}^0/_0 \text{ per year.}$$

The observations of KLEIN as well as those of PERETTI (fig. 5) give values of 2—3 % per annum for the diminution of P with the time,

Fig. 4.



although it must be remarked that, according to PERETTI, for 6-year olds the reduction is practically nil after 1932.

From figs. 4 and 5 can also be deduced the value of $\frac{\partial P}{\partial l}$ i.e. the dependence of P on age.

Thus, for instance, in fig. 4 the line for $l = 13$ is about 10 % higher than that for $l = 10\frac{1}{2}$, from which there follows for $\frac{\partial P}{\partial l}$ a value of $\frac{10}{(13-10\frac{1}{2})} = 4$.

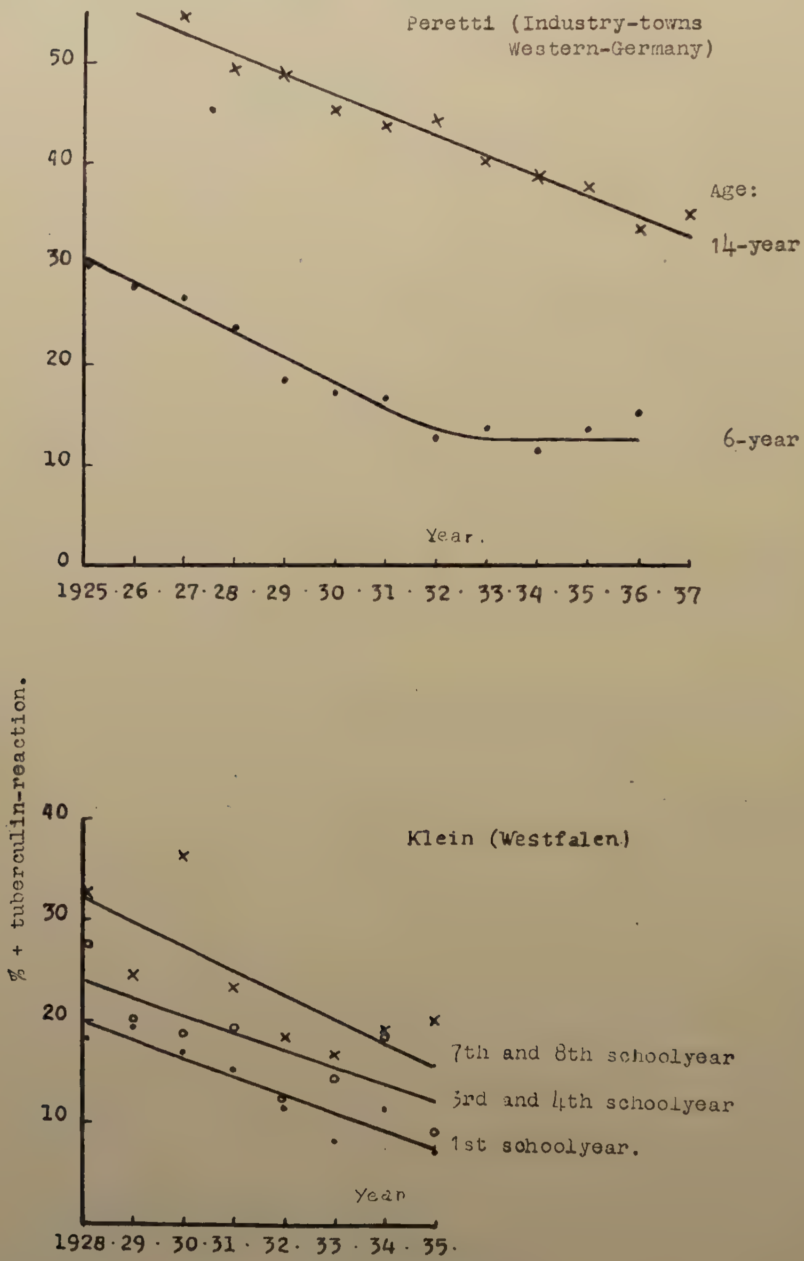
If this method of calculation is applied to the lines of figs. 4 and 5, we get as a mean value:

$$\frac{\partial P}{\partial l} = 3.5\ {}^0/_0 \text{ per year,}$$

a value which again applies for the approximately 10-year olds and for the year 1930.

It must not be forgotten here that the values of $\frac{\partial P}{\partial l}$, found from the different pairs of lines, differ considerably (from about 1.8 % to about

Fig. 5.

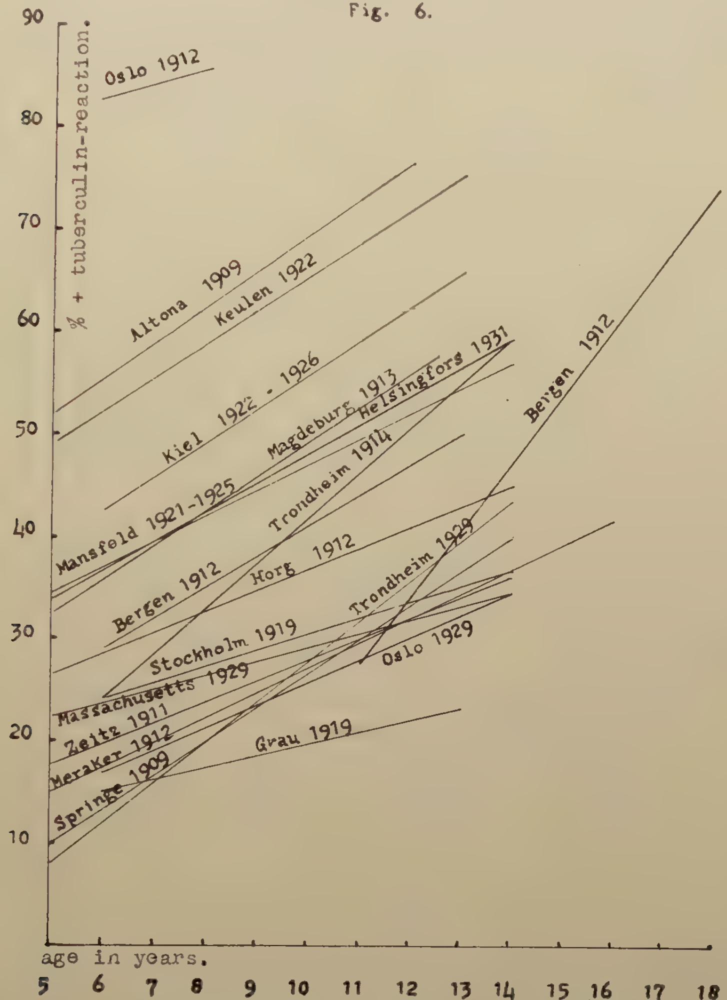


Dates taken from E.Schröder,
Ergebn.d.ges.Tbk-forschung.Bd VIII.

4.4 %), so that the above value of $\left(\frac{\partial P}{\partial l}\right)\begin{matrix} l=10 \\ t=1930 \end{matrix}$ must merely be considered as a rough estimation.

The great uncertainty appears very clearly from fig. 6, in which P is now represented as a function of the age l.

Fig. 6.



Dates taken from Mattison. Ergebn.d.ges.Tbk.forschung. Bd. V.

Here, with one exception, each line is established by only two points and therefore we cannot do otherwise as to draw a straight line between these points.

By their slope the lines give $\frac{\partial P}{\partial l}$ and the values for $\frac{\partial P}{\partial l}$ vary between 1.5 to 6.5 % per year. This does not conflict with the above-mentioned value of 3.5 % per annum, but is of little value next to the more reliable data.

Measurements of the same group of the population in different years only occur once here (TRONDHJEIM). This gives for $\frac{\partial P}{\partial l}$ a value of 1 % per year. However, not too much value must be attached to the latter, because it is only one of the many lines.

The rate of infection b for 10-year olds in 1930 for a number of large towns in Germany finally follows from equation 5, applied to the graphs of figs. 4 and 5:

$$b = \frac{1}{1-0.30}(3.5-2.5) = 1.5\ \%\ \text{per year.}$$

From this it can be seen that neglection of the term $\frac{\partial P}{\partial t}$, the moving up of the year groups, would give an entirely incorrect result of

$$b = \frac{1}{1-0.30} \times 3.5 = 5\ \%\ \text{per year.}$$

Actually, therefore, the consideration of MUENCH and of HEIJNSIUS VAN DEN BERG, who only take into account the dependence of P on the age l, is not only fundamentally, but also quantitatively, incorrect, as was already expected at the beginning of this article on the basis of more qualitative considerations.

There are of course many cases where the calculation of HEIJNSIUS VAN DEN BERG gives a good approximation. If the figures for TRONDHJEIM may be relied upon, it then follows from them that $\frac{\partial P}{\partial l} = 4.5\ \%$ per year and $\frac{\partial P}{\partial l} = 1\ \%$ per year, therefore $b = \frac{1}{1-0.3}(4.5-1) = 5\ \%$ per year (for 10-year olds in the period 1914—1929).

Here omission of the term $\frac{\partial P}{\partial l}$, would give a fairly good result of $\frac{1}{1-0.30} \times 4.5 = 6.5\ \%$ per year.

It is also possible to endeavour to arrive at an estimation of the rate of infection for other ages:

$$b_{l=12} = \frac{1}{1-0.35}(4-2.8) = 2\ \%\ \text{per year,}$$

$$b_{l=8} = \frac{1}{1-0.25}(3.5-2.2) = 1.7\ \%\ \text{per year}^{1)}.$$

It is doubtful whether the somewhat higher value that is found for the rate of infection at older ages has a real significance. It is also certain that discarding $\frac{\partial P}{\partial t}$ would give an entirely incorrect result, as follows from

[1]) These figures have been deduced from SCHRÖDER's observations only (fig. 4), which is why the values are slightly larger than those for 10-year olds mentioned above.

the calculation by HEIJNSIUS VAN DEN BERG, who found large differences in the rate of infection between these age groups.

Finally, as regards the variation of the infection rate with the time (year), it must be observed that the lines of fig. 4 come closer together towards the right. This means a diminution of $\frac{\partial P}{\partial l}$ with the time.

Assuming the lines to be straight, this would signify that $\frac{\partial P}{\partial t}$ is independent of the time. The result is consequently a reduction of $\frac{\partial P}{\partial l} + \frac{\partial P}{\partial t}$.

As, moreover, the risk of infection must be divided by $1 - P$, where P diminishes with the time, there follows from both facts a decrease of the rate of infection b with the time.

Finally, a further remark of a general nature, regarding the statistical problem dealt with here. Two entirely different questions can be asked. In the first place the important magnitudes (here the rate of infection: b can be deduced from the experimental data and equation (4) which are related to the mechanism of the phenomena.

In the second place an endeavour can be made, with the aid of the equations, to give a reply to the question as to what the future will bring us. This second method of dealing with the problem, which is a very usual one in physics, astronomy, chemistry, etc. encounters great difficulties when applied to "human" problems. One would have to forecast how in the future the rate of infection b would depend on time and age in order to subsequently "solve" equation (4). While this solution is of a purely mathematical nature and in principle simple and practicable in actual practice, the forecast is a medical problem of a less simple nature. Though it may certainly be said that b depends on the number of sources of infection and will be proportional to the same, the proportionality-factor depends on many social and hygienic factors (isolation of patients, succeptibility, etc.) and although we might dare to say something about them as regards the present time, an extrapolation for the future would be hazardous.

For this reason it would seem to us that only the evaluation from the available data of the rate of infection in the past and in the present is of any value. To make it possible in future to have correct data available for treating tuberculin statistics in the right way, it is necessary to establish the percentage of positive reactions for every calendar year and for every age group.

Summary.

From the statistics of the tuberculin reaction a closer insight can be obtained into the greatness of the rate of infection according to age and year. For the evaluation of this rate of infection a simple relation is deduced.

Mathematics. — *Beiträge zur Theorie der* WHITTAKER*schen Funktionen.* (Erste Mitteilung). Von C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of May 28, 1938.)

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§ 1. Bezeichnungen.

Es bezeichnen $J_\nu(z)$, $Y_\nu(z)$, $H_\nu^{(1)}(z)$ und $H_\nu^{(2)}(z)$ die BESSELschen und HANKELschen Funktionen. Es sei weiter

$$I_\nu(z) = e^{-\frac{1}{2}\nu\pi i} J_\nu(z e^{\frac{1}{2}\pi i}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

und

$$K_\nu(z) = \tfrac{1}{2}\pi i\, e^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}(z e^{\frac{1}{2}\pi i}). \quad \ldots \ldots \quad (2)$$

Es bezeichnen ferner $M_{k,m}(z)$ und $W_{k,m}(z)$ die beiden WHITTAKERschen Funktionen. D. h. also [1]):

Ist $z \neq 0$ und $2m \neq -1, -2, -3, \ldots$, so ist

$$M_{k,m}(z) = e^{-\frac{1}{2}z} z^{m+\frac{1}{2}}\, {}_1F_1(\tfrac{1}{2} - k + m; 2m+1; z); \quad \ldots \quad (3)$$

ist $z \neq 0$ und $2m$ nicht ganz, so ist [2])

$$W_{k,m}(z) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k-m)} M_{k,m}(z) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}-k+m)} M_{k,-m}(z). \quad . \quad (4)$$

Zur Abkürzung setze ich noch

$$T_{k,m}(z) = e^{\frac{1}{2}z^2} W_{k,m}(z^2) \quad \ldots \ldots \quad (5)$$

und

$$L_{k,m}(z) = \frac{e^{-\frac{1}{2}z^2}}{\Gamma(2m+1)} M_{k,m}(z^2), \quad \ldots \ldots \quad (6)$$

so dass aus (3) folgt

$$\left.\begin{aligned} L_{k,m}(z) = e^{-z^2} z^{2m+1} \Big\{ &\frac{1}{\Gamma(2m+1)} + \frac{\frac{1}{2}-k+m}{\Gamma(2m+2)} \frac{z^2}{1!} \\ &+ \frac{(\frac{1}{2}-k+m)(\frac{3}{2}-k+m)}{\Gamma(2m+3)} \frac{z^4}{2!} + \ldots \Big\}. \end{aligned}\right\} \quad . \quad . \quad (7)$$

Die rechterhand vorkommende Entwicklung hat auch einen Sinn für $2m = -1, -2, -3, \ldots$. Die Funktion $L_{k,m}(z)$ kann also für *alle* Werte von m durch (7) erklärt werden.

Schliesslich bezeichnet [3])

$$D_n(z) = 2^{\frac{1}{2}n+\frac{1}{4}} z^{-\frac{1}{2}} W_{\frac{1}{2}n+\frac{1}{4}, \pm\frac{1}{4}}(\tfrac{1}{2}z^2). \quad \ldots \ldots \quad (8)$$

die WEBERsche Funktion des parabolischen Zylinders.

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[1]) WHITTAKER and WATSON, [**38**], chapter XVI. (Die fett gedruckten Zahlen beziehen sich auf das Literaturverzeichnis).

Die Funktion $M_{k,m}(z)$ wird nicht definiert, falls $2m = -1, -2, -3, \ldots$ ist.

$${}_1F_1(\alpha; \beta; z) = 1 + \frac{\alpha}{\beta}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{z^2}{2!} + \ldots$$

(${}_pF_q(\alpha_1, \ldots, \alpha_p;\ \beta_1, \ldots, \beta_q;\ z)$ ist eine verallgemeinerte hypergeometrische Funktion).

[2]) Für ganzzahlige Werte von $2m$ kann $W_{k,m}(z)$ durch Grenzübergang definiert werden. Mann vergl. WHITTAKER and WATSON, [**38**], §§ 16.12 und 16.4; siehe auch MEIJER, [**20**], 35—36 und ERDÉLYI, [**10**], 347—348.

[3]) Man vergl. WHITTAKER and WATSON, [**38**], § 16.5. Die Funktion $W_{k,m}(z)$ ist eine gerade Funktion von m (siehe (4)).

§ 2. Zusammenfassung der Ergebnisse.

In einer vorigen Arbeit [4]) habe ich für die Funktion $T_{k,m}(z)$ die folgende Integraldarstellung abgeleitet:

$$T_{k,m}(z) = \frac{4z}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)} \int_0^{\infty e^{i\tau}} e^{-u^2} K_{2m}(2zu)\, u^{-2k}\, du. \quad (9)$$

Hierin ist $\Re(\frac{1}{2}-k \pm m) > 0$, z beliebig ($z \neq 0$) und τ ein Punkt des Intervalles [5])

$$-\tfrac{1}{4}\pi < \tau < \tfrac{1}{4}\pi. \quad \ldots \ldots \ldots \quad (10)$$

Eine verwandte Integralformel ist neuerdings von Herrn A. ERDÉLYI [6]) hergeleitet worden; er beweist

$$T_{k,m}(z) = e^{(m-k+\frac{1}{2})\pi i}\, z e^{z^2} \int_S e^{-u^2} H^{(1)}_{2m}(2zu)\, u^{2k}\, du. \quad \ldots \quad (11)$$

In dieser Beziehung sind k, m und z beliebig ($z \neq 0$); der Integrationsweg S läuft von $\infty e^{i(\tau+\pi)}$ nach $\infty e^{i\tau}$ (τ genügt der Bedingung (10)) und zwar so, dass der Punkt $u = 0$ durch einen oberhalb dieses Punktes liegenden Halbkreis vermieden wird.

In einer andern Arbeit gibt Herr ERDÉLYI [7]) eine analoge Integraldarstellung für $L_{k,m}(z)$, nämlich

$$L_{k,m}(z) = \frac{2z}{\Gamma(\frac{1}{2}+k+m)} \int_0^{\infty e^{i\tau}} e^{-u^2} J_{2m}(2zu)\, u^{2k}\, du. \quad \ldots \quad (12)$$

Hierin ist $\Re(\frac{1}{2}+k+m) > 0$, z beliebig ($z \neq 0$) und τ ein Punkt des Intervalles (10).

In der vorliegenden Abhandlung werde ich zeigen, dass die Beziehungen

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[4]) MEIJER, **[21]**, Formel (3).

[5]) In **[21]** gebe ich Formel (9) nur für $\tau = 0$. Es ist aber leicht einzusehen, dass das Integral in (9) sich nicht ändert, wenn der Integrationsweg durch die positive reelle Achse ersetzt wird (man vergl. die Beziehungen (77) und (79) von § 4; siehe auch Fussnote [45])).

[6]) ERDÉLYI, **[10]**, 348. Integraldarstellung (11) ist in etwas andrer Gestalt unter der Voraussetzung $\Re(\frac{1}{2}+k \pm m) > 0$ vom Verfasser (**[22]**, Formel (1)) abgeleitet worden.

[7]) ERDÉLYI, **[11]**, 359. ERDÉLYI gibt auch die Integralformel

$$L_{k,m}(z) = \frac{1}{\pi}\,\Gamma(\tfrac{1}{2}-k-m)\, e^{-(k+m)\pi i}\, z \int_S e^{-u^2} J_{2m}(2zu)\, u^{2k}\, du,$$

wobei die Voraussetzung $\Re(\frac{1}{2}+k+m) > 0$ fortgelassen werden darf (S hat dieselbe Gestalt wie in (11)). Diese Beziehung ist äquivalent mit (12). Denn es gilt (siehe ERDÉLYI, **[10]**, 350)

$$\int_0^{\infty} e^{-u^2} J_{2m}(2zu)\, u^{2k}\, du = \frac{e^{-(k+m)\pi i}}{2\cos(k+m)\pi} \int_S e^{-u^2} J_{2m}(2zu)\, u^{2k}\, du.$$

(9), (11) und (12) nur Spezialfälle viel allgemeinerer Integraldarstellungen sind. Ich werde nämlich — *und zwar auf möglichst elementare Weise* — die folgenden Sätze beweisen [8]):

Satz 1. *Ist* $z \neq 0$,

$$|\arg z| < \tfrac{3}{4}\pi, \qquad (13)$$

$$\Re\left(\tfrac{1}{2} - k + m\right) > 0, \qquad (14)$$

$$\Re(2 - k - 3m) > 0 \qquad (15)$$

und genügt τ *den Bedingungen*

$$-\tfrac{1}{4}\pi < \tau < \tfrac{1}{4}\pi, \qquad (16)$$

$$-\tfrac{1}{2}\pi - \arg z \leqq \tau \leqq \tfrac{1}{2}\pi - \arg z, \qquad (17)$$

so gilt

$$T_{k,m}(z) = \frac{4 z^{m-k+2\alpha+3/2}}{\Gamma\left(\frac{1}{2} - k + m\right)} \int_0^{\infty e^{i\tau}} L_{-m-\alpha,\alpha-k}(u)\, K_{m+k-2\alpha-\frac{1}{2}}(2zu)\, u^{m-k-\frac{1}{2}}\, du \qquad (18)$$

Hierin ist α *eine beliebige Zahl mit* [9])

$$\Re(1 - 2k + 2\alpha) > 0, \qquad (19)$$

$$\Re(1 + k - 3m - 2\alpha) > 0 \qquad (20)$$

Bemerkung 1. *Gelten* (13), (14), (19), (16) *und* (17), *wobei in* (17) *die beiden Zeichen* $\leqq$ *durch* $<$ *ersetzt werden, so gilt* (18) *gleichfalls; die Bedingungen* (15) *und* (20) *dürfen dann also fortgelassen werden.*

Bemerkung 2. *Gelten* (14) *und* (19) *und ist ausserdem* $k - m - 2\alpha = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, *so gilt Beziehung* (18) *für jedes* $z \neq 0$ *und jedes* τ *mit* (16); *die Bedingungen* (13), (15), (20) *und* (17) *brauchen dann also nicht erfüllt zu sein.*

Satz 2. *Ist* $z \neq 0$, $|\arg z| < \frac{3}{4}\pi$ *und*

$$\Re\left(\tfrac{1}{2} - k + m\right) > 0, \qquad (21)$$

so gilt [10])

$$\left.\begin{aligned} T_{k,m}(z) &= \frac{2 z^{m-k+2\alpha+3/2}\, \Gamma\left(\frac{1}{2} + k - m - 2\alpha\right)}{\Gamma\left(\frac{1}{2} - k + m\right)} \\ &\times \int_0^{\infty e^{-i \arg z}} T_{m+\alpha,k-\alpha}(u)\, J_{m+k-2\alpha-\frac{1}{2}}(2zu)\, u^{m-k-\frac{1}{2}}\, du. \end{aligned}\right\} \qquad (22)$$

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[8]) Mittels $T_{k,m}(z) = T_{k,-m}(z)$ kann man aus den Sätzen 1—4 noch andre Integraldarstellungen für $T_{k,m}(z)$ ableiten.

[9]) Die Ungleichungen (19) und (20) sind lösbar; denn aus (15) folgt $\Re(1 + k - 3m) > \Re(2k - 1)$.

[10]) Relation (22) kommt mit ganz andern Bezeichnungen, und allein für positive Werte von z, schon bei Herrn BAILEY ([3], 264, Formel (7.2)) vor. BAILEY gibt (22) ohne Beweis; er beweist nur den Spezialfall mit $\alpha = k - m$. Ein andrer Sonderfall von (22) ist neulich von R. S. VARMA ([36], 211) abgeleitet worden. Mein Beweis von Formel (22) ist demjenigen des BAILEYschen Spezialfalles nahe verwandt.

Hierin ist a eine beliebige Zahl mit [11])

$$\Re\left(\tfrac{1}{2}+k+m-2a\right)>0, \qquad (23)$$

$$\Re\left(1+k-3m-2a\right)>0, \qquad (24)$$

$$\tfrac{1}{2}+k-m-2a\neq 0. \qquad (25)$$

Satz 3. *Ist* $z\neq 0$ *und*

$$|\arg z|<\tfrac{1}{4}\pi, \qquad (26)$$

so gilt

$$\left.\begin{aligned} T_{k,m}(z) &= 2\,z^{m+k+2\alpha+3/2}\,e^{z^2}\,\Gamma\left(\tfrac{1}{2}-k-m-2\alpha\right) \\ &\times \int_0^{\infty e^{i\tau}} L_{-m-\alpha,-k-\alpha}(u)\,J_{m-k-2\alpha-\frac{1}{2}}(2zu)\,u^{m+k-\frac{1}{2}}\,du. \end{aligned}\right\} \qquad (27)$$

Hierin ist

$$\tau=-\arg z \qquad (28)$$

und a eine beliebige Zahl mit

$$\Re\left(\tfrac{1}{2}-k+m-2a\right)>0, \qquad (29)$$

$$\Re\left(1-k-3m-2a\right)>0, \qquad (30)$$

$$\tfrac{1}{2}-k-m-2a\neq 0 \qquad (31)$$

Bemerkung. *Gilt* (29) *und ist* $\tfrac{1}{2}-k-m-2\alpha\neq 0,-1,-2,\ldots$ *und* $k-m=\tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2},\ldots,$ *so gilt Beziehung* (27) *für jedes* $z\neq 0$ *und jedes* τ *mit* $-\tfrac{1}{4}\pi<\tau<\tfrac{1}{4}\pi$; *die Bedingungen* (26), (28) *und* (30) *brauchen dann also nicht erfüllt zu sein.*

Satz 4. *Ist* $z\neq 0$,

$$|\arg z|<\tfrac{1}{4}\pi. \qquad (32)$$

und

$$m-k\neq\tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2},\ldots, \qquad (33)$$

so gilt

$$\left.\begin{aligned} T_{k,m}(z) &= e^{-(2k+2\alpha)\pi i}\,z^{m+k+2\alpha+3/2}\,e^{z^2}\,\Gamma\left(\tfrac{1}{2}+k-m\right) \\ &\times \int_S L_{-m-\alpha,k+\alpha}(u)\,H^{(1)}_{m-k-2\alpha-\frac{1}{2}}(2zu)\,u^{m+k-\frac{1}{2}}\,du. \end{aligned}\right\} \qquad (34)$$

Hierin ist a eine beliebige Zahl mit

$$\Re\left(1-k-3m-2a\right)>0 \qquad (35)$$

Der Integrationsweg S läuft von $\infty\, e^{i(\pi-\arg z)}$ *nach* $\infty\, e^{-i\arg z}$ *und zwar*

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[11]) Aus (23) und (24) folgt $\Re\left(\tfrac{3}{4}+k-m-2\alpha\right)>0$; mit Rücksicht auf (25) gilt also $\tfrac{1}{2}+k-m-2\alpha\neq 0,-1,-2,\ldots,$ so dass die in (22) vorkommende Funktion $\Gamma\left(\tfrac{1}{2}+k-m-2\alpha\right)$ einen Sinn hat.

so, dass der Punkt $u = 0$ durch einen oberhalb dieses Punktes liegenden Halbkreis vermieden wird.

Bemerkung. *Gilt* (33) *und ist ausserdem* $m+k+2\alpha = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$, *so gilt Beziehung* (34) *für jedes* $z \neq 0$ *(die Bedingungen* (32) *und* (35) *brauchen dann also nicht erfüllt zu sein). Der Integrationsweg S läuft von* $\infty e^{i(\tau+\pi)}$ *nach* $\infty e^{i\tau}$ (τ *ist eine beliebige Zahl mit* $-\frac{1}{4}\pi < \tau < \frac{1}{4}\pi$) *und hat dieselbe Gestalt wie in* (11).

Satz 5. *Ist* $z \neq 0$,

$$|\arg z| < \tfrac{1}{4}\pi \quad \ldots\ldots\ldots\ldots \quad (36)$$

und

$$\Re(2+k+3m) > 0, \quad \ldots\ldots\ldots \quad (37)$$

so gilt

$$L_{k,m}(z) = 2 z^{k-m+2\alpha+3/2} \int_0^{\infty e^{i\tau}} L_{m-\alpha,k+\alpha}(u) J_{m+k+2\alpha+\frac{1}{2}}(2zu)\, u^{k-m-\frac{1}{2}}\, du. \quad (38)$$

Hierin ist

$$\tau = -\arg z \quad \ldots\ldots\ldots\ldots \quad (39)$$

und α eine beliebige Zahl mit

$$\Re(1+2k+2\alpha) > 0, \quad \ldots\ldots\ldots \quad (40)$$

$$\Re(1-k+3m-2\alpha) > 0. \quad \ldots\ldots\ldots \quad (41)$$

Bemerkung. *Gilt* (40) *und ist ausserdem* $m-k-2\alpha = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, *so gilt Beziehung* (38) *für jedes* $z \neq 0$ *und jedes* τ *mit* $-\frac{1}{4}\pi < \tau < \frac{1}{4}\pi$; *die Bedingungen* (36), (39), (37) *und* (41) *brauchen dann also nicht erfüllt zu sein.*

Satz 6. *Ist* $\Re\,(\frac{1}{2}+k+m) > 0$ *und* $-\frac{1}{4}\pi < \tau < \frac{1}{4}\pi$, *so gilt für jedes* $z \neq 0$

$$L_{k,m}(z) = \frac{2 z^{m+k+2\alpha+3/2}}{\Gamma(\frac{1}{2}+k+m)} \int_0^{\infty e^{i\tau}} e^{-u^2}\, T_{-m-\alpha,k+\alpha}(u) J_{m-k-2\alpha-\frac{1}{2}}(2zu)\, u^{m+k-\frac{1}{2}}\, du. \quad (42)$$

Hierin ist α eine beliebige Zahl mit $\Re\,(\frac{1}{2}-k+m-2\alpha) > 0$.

Satz 7. *Ist* $\Re\,(\frac{1}{2}-k+m) > 0$ *und* $-\frac{1}{4}\pi < \tau < \frac{1}{4}\pi$, *so gilt für jedes* $z \neq 0$

$$L_{k,m}(z) = \frac{2 z^{m-k+2\alpha+3/2} e^{-z^2}}{\Gamma(\frac{1}{2}-k+m)} \int_0^{\infty e^{i\tau}} e^{-u^2}\, T_{-m-\alpha,-k+\alpha}(u)\, I_{m+k-2\alpha-\frac{1}{2}}(2zu)\, u^{m-k-\frac{1}{2}}\, du. \quad (43)$$

Hierin ist α eine beliebige Zahl mit $\Re\,(\frac{1}{2}+k+m-2\alpha) > 0$.

Ich werde jetzt zeigen, dass die Formeln (9), (11) und (12) nur Spezialfälle der vorangehenden Bemerkungen sind. Aus (7) folgt nämlich

$$L_{\lambda,\lambda-\frac{1}{2}}(u) = \frac{e^{-u^2} u^{2\lambda}}{\Gamma(2\lambda)}.$$

Beziehung (9) ist also der Spezialfall mit $2\alpha = k - m - \frac{1}{2}$ von (18) (Bemerkung 2). Relation (11) der Spezialfall mit $2\alpha = -m - k - \frac{1}{2}$ von (34) (siehe die Bemerkung zu Satz 4)[12]) und Integraldarstellung (12) der Spezialfall mit $2\alpha = m - k - \frac{1}{2}$ von (38) (siehe die Bemerkung zu Satz 5).

Aus (5), (4) und (3) geht hervor

$$T_{\lambda, \lambda - \frac{1}{2}}(u) = u^{2\lambda}, \quad \ldots \ldots \ldots \quad (44)$$

so dass Beziehung (12) auch ein Sonderfall, nämlich mit $2\alpha = -m - k - \frac{1}{2}$, von (42) ist[13]).

Der Spezialfall mit $2\alpha = k - m - \frac{1}{2}$ von (43) ergibt wegen (44)

$$L_{k,m}(z) = \frac{2 z e^{-z^2}}{\Gamma(\frac{1}{2} - k + m)} \int_0^{\infty e^{i\tau}} e^{-u^2} I_{2m}(2zu)\, u^{-2k}\, du.$$

Diese Integralformel gilt für alle Werte von k und m mit $\Re(\frac{1}{2} - k + m) > 0$.

In § 3 gebe ich einige andre Sonderfälle der vorangehenden Sätze[14]) und Anwendungen auf selbstreziproke Funktionen. Ferner werde ich noch die folgenden Sätze beweisen:

Satz 8. *Ist $z \neq 0$, $|\arg z| \leqq \frac{1}{2}\pi$ und $\Re(\frac{1}{2} - k \pm m) > 0$, so gilt*

$$\left.\begin{aligned} W_{k,m}(2z^2 e^{\frac{1}{2}\pi i})\, W_{k,m}(2z^2 e^{-\frac{1}{2}\pi i}) &= \frac{16\pi z^2}{\Gamma(\frac{1}{2} - k + m)\,\Gamma(\frac{1}{2} - k - m)} \\ &\times \int_0^\infty J_{m-k}(u^2)\, J_{-m-k}(u^2)\, K_{4m}(4zu)\, u\, du. \end{aligned}\right\} \quad (45)$$

Satz 9. *Ist $z > 0$, $\Re(m) > -\frac{1}{4}$ und $\Re(\frac{1}{2} \pm k + m) > 0$, so gilt*

$$\left.\begin{aligned} M_{k,m}(2z^2 e^{\frac{1}{2}\pi i})\, M_{k,m}(2z^2 e^{-\frac{1}{2}\pi i}) &= \frac{2\pi i z^2\, \Gamma^2(2m+1)}{\Gamma(\frac{1}{2} - k + m)\,\Gamma(\frac{1}{2} + k + m)} \\ \times \int_0^\infty \{H^{(1)}_{m+k}(u^2)\, H^{(1)}_{m-k}(u^2) &- H^{(2)}_{m+k}(u^2)\, H^{(2)}_{m-k}(u^2)\}\, J_{4m}(4zu)\, u\, du. \end{aligned}\right\} \quad (46)$$

---

12) Die Bedingung $m - k \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ kann in (11) durch Grenzübergang beseitigt werden.

13) Nur falls $\Re(\frac{1}{2} + k + m)$ und $\Re(2m + 1) > 0$ sind, folgt Formel (12) aus (42) (mit $2\alpha = -m - k - \frac{1}{2}$ angewendet). Das Integral in (12) konvergiert aber für $\Re(\frac{1}{2} + k + m) > 0$ und ist unter dieser Bedingung eine analytische Funktion von k und m. Durch analytische Fortsetzung schliesst man also, dass (12) für alle Werte von k und m mit $\Re(\frac{1}{2} + k + m) > 0$ gültig ist.

14) Spezialfälle der Sätze 3 und 6 sind vor kurzem auch von ERDÉLYI ([12], Formeln (5.5) und (5.6)) abgeleitet worden.

Satz 10. *Ist* $z \neq 0$, $|\arg z| \leqq \frac{1}{4}\pi$, $\Re(m) > -\frac{1}{4}$ *und* $\Re(\frac{1}{2}-k+m) > 0$, *so gilt*

$$W_{k,m}(2z^2)M_{-k,m}(2z^2) = \frac{8z^2\Gamma(2m+1)}{\Gamma(\frac{1}{2}-k+m)} \int_0^{\infty e^{-i\arg z}} K_{m+k}(u^2) I_{m-k}(u^2) J_{4m}(4zu)\,u\,du. \quad (47)$$

Satz 11. *Ist* $z > 0$ *und* $\Re(m) > -\frac{1}{4}$, *so gilt*

$$\left.\begin{aligned} e^{m\pi i} W_{k,m}(2z^2 e^{\frac{1}{2}\pi i}) W_{-k,m}(2z^2 e^{\frac{1}{2}\pi i}) + e^{-m\pi i} W_{k,m}(2z^2 e^{-\frac{1}{2}\pi i}) W_{-k,m}(2z^2 e^{-\frac{1}{2}\pi i}) \\ = 8\pi z^2 \int_0^{\infty} J_{m+k}(u^2) J_{m-k}(u^2) J_{4m}(4zu)\,u\,du. \end{aligned}\right\} \quad (48)$$

Der Spezialfall mit $m = k = 0$ von (47) war schon bekannt [15]).

§ 3. Anwendungen.

Die BESSELschen und HANKELschen Funktionen sind bekanntlich Spezialfälle der WHITTAKERschen Funktionen. Denn man hat [16])

$$J_\nu(\zeta) = \frac{e^{-\frac{1}{2}(\nu+\frac{1}{2})\pi i}}{2^{2\nu+\frac{1}{2}}\zeta^{\frac{1}{2}}\Gamma(\nu+1)} M_{0,\nu}(2\zeta e^{\frac{1}{2}\pi i}),$$

$$I_\nu(\zeta) = \frac{1}{2^{2\nu+\frac{1}{2}}\zeta^{\frac{1}{2}}\Gamma(\nu+1)} M_{0,\nu}(2\zeta), \quad \ldots \ldots \quad (49)$$

$$H_\nu^{(1)}(\zeta) = \left(\frac{2}{\pi\zeta}\right)^{\frac{1}{2}} e^{-\frac{1}{2}(\nu+\frac{1}{2})\pi i} W_{0,\nu}(2\zeta e^{-\frac{1}{2}\pi i})$$

und

$$K_\nu(\zeta) = \left(\frac{\pi}{2\zeta}\right)^{\frac{1}{2}} W_{0,\nu}(2\zeta) \quad \ldots \ldots \ldots \quad (50)$$

Wegen [17])

$$J_\nu(\zeta) = e^{\nu\pi i} J_\nu(\zeta e^{-\pi i}) \quad \ldots \ldots \ldots \quad (51)$$

und

$$H_\nu^{(2)}(\zeta) = -e^{\nu\pi i} H_\nu^{(1)}(\zeta e^{\pi i}). \quad \ldots \ldots \ldots \quad (52)$$

---

15) Formel (60) mit $\nu = 0$.

Herr ERDÉLYI hat mir geschrieben, dass die Beziehungen (47) und (60) auch vorkommen in einer Arbeit, die er demnächst in dem Journal of the London Mathematical Society veröffentlichen wird.

16) WHITTAKER and WATSON, [38], § 17.212; MEIJER, [20], Formel (17); [21], Formel (8).

17) WATSON, [37], 75, Formel (1) und Formel (5) mit $m = 1$.

gilt also

$$J_\nu^2(\zeta) = e^{\nu\pi i} J_\nu(\zeta) J_\nu(\zeta e^{-\pi i}) = \frac{1}{2^{4\nu+1}\zeta\, \Gamma^2(\nu+1)} M_{0,\nu}(2\zeta e^{\frac{1}{2}\pi i}) M_{0,\nu}(2\zeta e^{-\frac{1}{2}\pi i}), \quad (53)$$

$$H_\nu^{(1)}(\zeta) H_\nu^{(2)}(\zeta) = -e^{\nu\pi i} H_\nu^{(1)}(\zeta) H_\nu^{(1)}(\zeta e^{\pi i}) = \frac{2}{\pi\zeta} W_{0,\nu}(2\zeta e^{\frac{1}{2}\pi i}) W_{0,\nu}(2\zeta e^{-\frac{1}{2}\pi i}) \quad (54)$$

und [18])

$$\left.\begin{aligned} J_\nu(\zeta) Y_\nu(\zeta) &= -\frac{1}{4i}\left[\{H_\nu^{(2)}(\zeta)\}^2 - \{H_\nu^{(1)}(\zeta)\}^2\right] \\ &= -\frac{1}{2\pi\zeta}\left[e^{\nu\pi i}\{W_{0,\nu}(2\zeta e^{\frac{1}{2}\pi i})\}^2 + e^{-\nu\pi i}\{W_{0,\nu}(2\zeta e^{-\frac{1}{2}\pi i})\}^2\right] \end{aligned}\right\} . \quad (55)$$

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[18]) $J_\nu(\zeta) = \frac{1}{2}\{H_\nu^{(1)}(\zeta) + H_\nu^{(2)}(\zeta)\}$, $\quad Y_\nu(\zeta) = \frac{1}{2i}\{H_\nu^{(1)}(\zeta) - H_\nu^{(2)}(\zeta)\}$.

Mathematics. — *A Theorem on inhomogeneous Diophantine Inequalities.* By KURT MAHLER. [1] (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of May 28, 1938.)

Two years ago (Math. Ann. 113, 1936, 398—415), KHINTCHINE proved that if the system of homogeneous linear inequalities

$$0 < x < \gamma t^n, \quad |x\theta_i - y_i| < \frac{1}{t} \qquad (i = 1, 2, \ldots, n),$$

where $\gamma = \gamma(\theta_1, \ldots, \theta_n) > 0$ does not depend on t, has no integer solution for any $t > 1$, then the system of inhomogeneous linear inequalities

$$0 < x < \Gamma t^n, \quad |x\theta_i - y_i - a_i| < \frac{1}{t} \quad (i = 1, 2, \ldots, n),$$

where $\Gamma = \Gamma(\gamma, \theta_1, \ldots, \theta_n) > 0$ does not depend on t and $a_1, \ldots, a_n$, has an integer solution for all $t > 1$. MORDELL generalized this result and at the same time gave a simpler proof (Journal London Math. Soc. 12, 1937, 34—36 and 166—167). In this note, I shall prove a still more general theorem (Theorem 2), which contains the results of KHINTCHINE and MORDELL as special cases, while its proof remains nearly as simple as that of MORDELL.

1. Let $F(x_1, \ldots, x_n)$ be a real function of the n real variables $x_1, \ldots, x_n$ with the following properties:

(1): $F(0, \ldots, 0) = 0$, but $F(x_1, \ldots, x_n) > 0$ for $\sum_{k=1}^{n} x_k^2 > 0$.

(2): $F(tx_1, \ldots, tx_n) = |t| F(x_1, \ldots, x_n)$ for all real values of t.

(3): $F(x_1 + y_1, \ldots, x_n + y_n) \leqq F(x_1, \ldots, x_n) + F(y_1, \ldots, y_n)$.

(4): The convex body defined by the inequality $F(x_1, \ldots, x_n) \leqq 1$ has the volume I.

Then, by a theorem of MINKOWSKI (Geometrie der Zahlen, p. 218), there exist n^2 integers $X_k^{(l)}$ with determinant

$$d = |X_k^{(l)}|_{k,l=1,2,\ldots,n} \neq 0,$$

such that

$$\prod_{l=1}^{n} F(X_1^{(l)}, \ldots, X_n^{(l)}) \leqq \frac{2^n}{I}. \tag{5}$$

[1] I wish to express my thanks to Prof. MORDELL for his help with the manuscript.

MINKOWSKI also proved (G. d. Z., p. 189 and 192) that $|d| \leq n!$ and in particular $|d|=1$ for $n=2$; in general, however, $|d|=1$ need not be true. But the following weaker theorem holds:

Theorem 1: *Under the conditions* (1)—(4), *there are* n^2 *integers* $x_{hk}^{(l)}$ *with determinant*

$$|x_k^{(l)}|_{k,l=1,2,\ldots,n} = 1,$$

such that

$$(6): \quad \prod_{l=1}^{n} F(x_1^{(l)},\ldots,x_n^{(l)}) \leq \frac{2^n n!}{I}.$$

Proof: Suppose that $X_k^{(l)}$ are MINKOWSKI's integers, and that, if

$$F(X_1^{(l)},\ldots,X_n^{(l)}) = S_l \qquad (l=1,2,\ldots,n),$$

then without loss of generality

$$(7): \quad S_1 \leq S_2 \leq \ldots \leq S_n.$$

By MINKOWSKI's method of "adaptation" of the lattice of all points $(x_1,\ldots,x_n)$ with respect to the n lattice points $(X_1^{(l)},\ldots,X_n^{(l)})$ $(l=1,\ldots,n)$ (G. d. Z., p. 173—176), n lattice points $(x_1^{(l)},\ldots,x_n^{(l)})$ $(l=1,\ldots,n)$ with determinant

$$|x_k^{(l)}|_{k,l=1,\ldots,n} = 1,$$

exist, such that in vector notation for $l=1,\ldots,n$

$$(x_1^{(l)},\ldots,x_n^{(l)}) = \sum_{k=1}^{l} \beta_k^{(l)} (X_1^{(k)},\ldots,X_n^{(k)}),$$

where the $\beta_k^{(l)}$ are real numbers satisfying

$$|\beta_k^{(l)}| \leq 1 \qquad (k,l=1,\ldots,n;\ k \leq l).$$

Hence by (2), (3) and (7)

$$F(x_1^{(l)},\ldots,x_n^{(l)}) \leq S_1+S_2+\ldots+S_l \leq lS_l \leq lF(X_1^{(l)},\ldots,X_n^{(l)}),$$

so that (6) follows at once from (5).

2. From the last result we shall obtain:

Theorem 2: *Suppose that the conditions* (1)—(4) *are satisfied, that* $\xi_1,\ldots,\xi_n$ *are real numbers, that* τ *is a positive number, and that there is no other integer solution of the inequality*

$$(8): \quad F(X_1,\ldots,X_n) \leq \frac{2\tau}{\sqrt[n]{I}},$$

than the trivial one $X_1 = \ldots = X_n = 0$. *Then the inequality*

(9): $$F(x_1+\xi_1, \ldots, x_n+\xi_n) \leqslant \frac{(n+1)!+1}{\tau^{n-1}\sqrt[n]{I}}$$

has a solution in integers $x_1, \ldots x_n$.

Proof: Let $X_1, \ldots, X_n, Y$ be $n+1$ variables and consider the domain in $n+1$ dimensions defined by

(10): $$F(X_1+\xi_1 Y, \ldots, X_n+\xi_n Y) \leqslant \frac{2\tau}{\sqrt[n]{I}}, \quad |Y| \leqslant \frac{1}{\tau^n}.$$

It is easy to see that this domain is a convex body of volume

$$I\left(\frac{2\tau}{\sqrt[n]{I}}\right)^n \cdot \frac{2}{\tau^n} = 2^{n+1}.$$

Hence there are $n+1$ integers $x_1^*, \ldots, x_n^*, y$, which are not all zero (G. d. Z., p. 76), such that

(11): $$F(x_1^*+\xi_1 y, \ldots, x_n^*+\xi_n y) \leqslant \frac{2\tau}{\sqrt[n]{I}}, \quad |y| \leqslant \frac{1}{\tau^n}.$$

Here $y \neq 0$, since otherwise there would be a non-trivial integer solution $x_1, \ldots, x_n$ of (8), against hypothesis.

By theorem 1, there is a system of n^2 integers $x_k^{(l)}$ of determinant 1 satisfying (6). Obviously the numbers $x_1^{(l)}, \ldots, x_n^{(l)}$ do not vanish simultaneously for any l; hence by assumption

$$F(x_1^{(l)}, \ldots, x_n^{(l)}) > \frac{2\tau}{\sqrt[n]{I}} \qquad (l = 1, \ldots, n)$$

and therefore by (6)

(12): $$F(x_1^{(l)}, \ldots, x_n^{(l)}) \leqslant \frac{2n!}{\tau^{n-1}\sqrt[n]{I}} \qquad (l = 1, \ldots, n).$$

Now consider the system of n linear congruences in $u_1, \ldots, u_n$:

$$x_k^* + \sum_{l=1}^{n} x_k^{(l)} u_l \equiv 0 \ (mod\ y) \qquad (k = 1, \ldots, n).$$

Since its determinant is 1, there is at least one integer solution $u_1^*, \ldots, u_n^*$, and then all integer solutions can be written as

$$u_l = u_l^* + y v_l \qquad (l = 1, \ldots, n),$$

where $v_1, \ldots, v_n$ are arbitrary integers. Hence we may assume that

$$|u_l| \leq \frac{|y|}{2} \qquad (l = 1, \ldots, n).$$

With these values of the u's, put

$$x_k = \frac{1}{y}\left\{x_k^* + \sum_{l=1}^{n} x_k^{(l)} u_l\right\} \qquad (k = 1, \ldots, n),$$

so that $x_1, \ldots, x_n$ are integers and

$$x_k + \xi_k = \frac{1}{y}(x_k^* + y\,\xi_k) + \sum_{l=1}^{n} \frac{u_l}{y} x_k^{(l)},$$

and since $|y| \geq 1$,

$$F(x_1 + \xi_1, \ldots, x_n + \xi_n) \leq \frac{1}{|y|} F(x_1^* + \xi_1 y, \ldots, x_n^* + \xi_n y) + \sum_{l=1}^{n} \left|\frac{u_l}{y}\right| F(x_1^{(l)}, \ldots, x_n^{(l)})$$

$$\leq \frac{2\tau}{\sqrt[n]{I}} + \frac{n}{2} \cdot \frac{2n!}{\tau^{n-1}\sqrt[n]{I}} \leq \frac{2 + n.n!}{\tau^{n-1}\sqrt[n]{I}} \leq \frac{(n+1)!+1}{\tau^{n-1}\sqrt[n]{I}},$$

since by MINKOWSKI's theorem necessarily

$$\tau < 1,$$

q. e. d.

The example $F(x_1, \ldots, x_n) = max(\tau|x_1|, \tau|x_2|, \ldots, \tau|x_{n-1}|, \tau^{-(n-1)}|x_n|)$, $\xi_1 = \ldots = \xi_n = \frac{1}{2}$ shows, that the exponent $n-1$ of τ in theorem 2 cannot be improved.

Manchester, Mathematical Department of the University.

Eastern 1938.

Mathematics. — *Ueber Differentialkovarianten erster Ordnung der binären kubischen Differentialform.* Von P. G. MOLENAAR. (Communicated by Prof. R. WEITZENBÖCK.)

(Communicated at the meeting of May 28, 1938.)

§ 1. Die binäre kubische Differentialform

$$f = a_{ikl}\, dx^i\, dx^k\, dx^l = a^3_{dx}$$

hat eine Differentialkovariante erster Ordnung und zweiter Stufe mit den Komponenten

$$n_{ik} = \frac{\partial a_{ik1}}{\partial x_2} - \frac{\partial a_{ik2}}{\partial x_1} + \tfrac{1}{2}\left(\frac{\partial \alpha^{pq}}{\partial x_i}\, Q_{kpq} + \frac{\partial \alpha^{pq}}{\partial x_k}\, Q_{ipq}\right) \qquad (R \neq 0). \qquad [1]$$

Wegen der Relation

$$\alpha^{pq}\, Q_{pq\lambda} \equiv 0 \quad (\lambda = 1, 2) \quad \ldots \ldots \ldots \quad (1)$$

ist

$$n_{ik} = \frac{\partial a_{ik1}}{\partial x_2} - \frac{\partial a_{ik2}}{\partial x_1} - \tfrac{1}{2}\, \alpha^{pq}\left(\frac{\partial Q_{kpq}}{\partial x_i} + \frac{\partial Q_{ipq}}{\partial x_k}\right)$$

und da

$$\alpha^{pq} = \frac{2}{R}\, \alpha_{\underline{pq}}$$

findet man

$$n_{ik} = \frac{\partial a_{ik1}}{\partial x_2} - \frac{\partial a_{ik2}}{\partial x_1} - \frac{\alpha_{\underline{pq}}}{R}\left(\frac{\partial Q_{kpq}}{\partial x_i} + \frac{\partial Q_{ipq}}{\partial x_k}\right) \quad \ldots \quad (2)$$

Nun ist

$$\triangle = \alpha^2_{dx} = (f, f)^{(2)} \quad Q = (f, \triangle)^{(1)} \quad \text{und} \quad R = (\triangle, \triangle)^{(2)}. \quad \ldots \quad (3)$$

Durch (2) und (3) wird also eine Operation

$$n = N(f)$$

definiert, welche aus der Grundform f eine Differentialkovariante n ableitet.

Diese Operation kann man auch ausüben auf die absolute Differentialkovariante

$$Q^*_{ikl} = \frac{Q_{ikl}}{\sqrt{\frac{-R}{2}}} \quad \ldots \ldots \ldots \ldots \quad (4)$$

[1]) Vgl. P. G. MOLENAAR, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **41**, 278—288 (1938).

Hierzu muss man erst

$$\triangle_{Q^*},\quad R_{Q^*}\quad \text{und}\quad Q_{Q^*}$$

berechnen.

Das Büschel

$$f_{\varkappa\lambda} = \varkappa f + \lambda Q$$

hat die Komitanten [2])

$$\left.\begin{aligned}
\triangle_{\varkappa\lambda} &= \left(\varkappa^2 + \frac{R}{2}\lambda^2\right)\triangle \\
R_{\varkappa\lambda} &= \left(\varkappa^2 + \frac{R}{2}\lambda^2\right)^2 R \\
Q_{\varkappa\lambda} &= \left(\varkappa^2 + \frac{R}{2}\lambda^2\right)\left(\varkappa Q - \frac{R}{2}\lambda f\right)
\end{aligned}\right\} \quad \ldots\ldots \quad (5)$$

Setzt man $\varkappa = 0$ und $\lambda = 1$, so findet man

$$\begin{aligned}
\triangle_Q &= \tfrac{1}{2} R \triangle \\
R_Q &= \tfrac{1}{4} R^3 \\
Q_Q &= -\tfrac{1}{4} R^2 f
\end{aligned}$$

woraus folgt

$$\triangle_{Q^*} = \frac{\frac{1}{2} R \triangle}{\sqrt{\frac{-R}{2}}^{\,2}} = -\triangle$$

$$R_{Q^*} = \frac{\frac{1}{4} R^3}{\sqrt{\frac{-R}{2}}^{\,4}} = R$$

$$Q_{Q^*} = \frac{-\frac{1}{4} R^2 f}{\sqrt{\frac{-R}{2}}^{\,3}} = -\sqrt{\frac{-R}{2}}\, f.$$

Daher ist

$$u_{ik} = N(Q^*)_{ik} = \frac{\partial Q^*_{ik1}}{\partial x_2} - \frac{\partial Q^*_{ik2}}{\partial x_1} + \frac{a_{pq}}{R}\left\{\frac{\partial\left(-\sqrt{\frac{-R}{2}}\, a_{kpq}\right)}{\partial x_i} + \frac{\partial\left(-\sqrt{\frac{-R}{2}}\, a_{ipq}\right)}{\partial x_k}\right\} =$$

$$= \frac{\partial Q^*_{ik1}}{\partial x_2} - \frac{\partial Q^*_{ik2}}{\partial x_1} + \frac{a_{pq}}{R}\left(-\sqrt{\frac{-R}{2}}\right)\left(\frac{\partial a_{kpq}}{\partial x_i} + \frac{\partial a_{ipq}}{\partial x_k}\right) -$$

$$- \frac{a_{pq}}{R}\left(a_{kpq}\frac{\partial\sqrt{\frac{-R}{2}}}{\partial x_i} + a_{ipq}\frac{\partial\sqrt{\frac{-R}{2}}}{\partial x_k}\right)$$

2) Vgl. Clebsch—Lindemann, Vorlesungen über Geometrie I. S. 227.

also wegen

$$\alpha_{pq}\, a_{pq\lambda} \equiv 0 \quad (\lambda = 1, 2) \quad \ldots \ldots \ldots \quad (6)$$

$$u_{ik} = \frac{\partial Q^*_{ik1}}{\partial x_2} - \frac{\partial Q^*_{ik2}}{\partial x_1} + \frac{\frac{1}{2}\alpha_{pq}}{\sqrt{-\frac{R}{2}}}\left(\frac{\partial a_{kpq}}{\partial x_i} + \frac{\partial a_{ipq}}{\partial x_k}\right).$$

Setzt man

$$\frac{\alpha_{pq}}{\sqrt{\frac{-R}{2}}} = \overset{*}{\alpha}_{pq} \quad \ldots \ldots \ldots \quad (7)$$

so wird

$$u_{ik} = \frac{\partial Q^*_{ik1}}{\partial x_2} - \frac{\partial Q^*_{ik2}}{\partial x_1} + \tfrac{1}{2}\overset{*}{\alpha}_{pq}\left(\frac{\partial a_{kpq}}{\partial x_i} + \frac{\partial a_{ipq}}{\partial x_k}\right) \quad \ldots \quad (8)$$

Ferner folgt aus (2) nach (4)

$$n_{ik} = \frac{\partial a_{ik1}}{\partial x_2} - \frac{\partial a_{ik2}}{\partial x_1} - \frac{\alpha_{pq}}{R}\left\{\frac{\partial\left(\sqrt{\frac{-R}{2}}\, Q^*_{kpq}\right)}{\partial x_i} + \frac{\partial\left(\sqrt{\frac{-R}{2}}\, Q^*_{ipq}\right)}{\partial x_k}\right\}$$

oder wegen (1) und (7)

$$n_{ik} = \frac{\partial a_{ik1}}{\partial x_2} - \frac{\partial a_{ik2}}{\partial x_1} + \tfrac{1}{2}\overset{*}{\alpha}_{pq}\left(\frac{\partial Q^*_{kpq}}{\partial x_i} + \frac{\partial Q^*_{ipq}}{\partial x_k}\right) \quad \ldots \quad (9)$$

Man kann zeigen, dass die Differentialkovariante u_{ik} durch Ueberschiebung von n_{ik} über α^*_{ik} entsteht.

Es gibt eine lineare Transformation [3], welche

$$f = a_{ikl}\, dx^i\, dx^k\, dx^l$$

überführt in

$$f = a_{111}\, dx'^3 + a_{222}\, dx''^3 \quad \ldots \ldots \quad (10$$

Durch diese Transformation wird

$$\begin{aligned} Q &= a^2_{111}\, a_{222}\, dx'^3 - a^2_{222}\, a_{111}\, dx''^3 \\ \Delta &= 2\, a_{111}\, a_{222}\, dx'\, dx'' \\ R &= -2\, a^2_{111}\, a^2_{222} \\ Q^* &= a_{111}\, dx'^3 - a_{222}\, dx''^3 \\ \Delta^* &= 2\, dx'\, dx'' \end{aligned}$$

$$n_{11} = \frac{\partial a_{111}}{\partial x_2} \qquad n_{12} = 0 \qquad n_{22} = -\frac{\partial a_{222}}{\partial x_1}$$

$$u_{11} = \frac{\partial a_{111}}{\partial x_2} \qquad u_{12} = 0 \qquad u_{22} = \frac{\partial a_{222}}{\partial x_1}$$

[3]) Vgl. GORDAN—KERSCHENSTEINER, Invariantentheorie II, S. 177.

Die Kovarianten n_{ik} und u_{ik} sind also verschieden.

Durch Ueberschiebung von n_{ik} über a^*_{ik} bekommt man

$$(n, a^*)^{(1)} = (n_{11} a^*_{12} - n_{12} a^*_{11})\, dx'^2 + (n_{11} a^*_{22} - n_{22} a^*_{11})\, dx^1 dx^2 + (n_{12} a^*_{22} - n_{22} a^*_{12})\, dx^{22}.$$

Durch die obige Transformation entsteht hieraus

$$(n, a^*)^{(1)} = n_{11}\, dx'^2 - n_{22}\, dx^{22} = u_{11}\, dx'^2 + u_{22}\, dx^{22}.$$

u_{ik} ist also die erste Ueberschiebung von n_{ik} über a^*_{ik}.

Da

$$Q^* = (f, a^*)^{(1)}$$

hat man die Relation

$$N((f, a^*)^{(1)}) = (N(f), a^*)^{(1)}$$

§ 2. Setzt man in dem Büschel

$$f_{\varkappa\lambda} = \varkappa f + \lambda Q$$

$$\lambda = \frac{1}{\sqrt{\frac{-R}{2}}} \quad \text{also} \quad \lambda Q = Q^*$$

so ist

$$f_\varkappa = \varkappa f + Q^*$$

eine absolute Differentialkovariante ($\varkappa$ ist unabhängig von x_i).

Aus (5) folgt

$$\Delta_\varkappa = (\varkappa^2 - 1)\,\Delta$$

$$R_\varkappa = (\varkappa^2 - 1)^2 R$$

$$Q_\varkappa = (\varkappa^2 - 1)\left(\varkappa Q + \sqrt{\frac{-R}{2}}\, f\right) = \sqrt{\frac{-R}{2}}\,(\varkappa^2 - 1)(\varkappa Q^* + f).$$

Durch Anwendung der Operation N auf $f_\varkappa$ entsteht

$$S^{(\varkappa)}_{ik} = \frac{\partial(\varkappa a_{ik1} + Q^*_{ik1})}{\partial x_2} - \frac{\partial(\varkappa a_{ik2} + Q^*_{ik2})}{\partial x_1} -$$

$$- \frac{(\varkappa^2 - 1)\, a_{pq}}{(\varkappa^2 - 1)^2 R} \left\{ \frac{\partial \sqrt{\frac{-R}{2}}\,(\varkappa^2 - 1)(\varkappa Q^*_{kpq} + a_{kpq})}{\partial x_i} + \frac{\partial \sqrt{\frac{-R}{2}}\,(\varkappa^2 - 1)(\varkappa Q^*_{ipq} + a_{ipq})}{\partial x_k} \right\} =$$

$$= \varkappa\left(\frac{\partial a_{ik1}}{\partial x_2} - \frac{\partial a_{ik2}}{\partial x_1}\right) + \left(\frac{\partial Q^*_{ik1}}{\partial x_2} - \frac{\partial Q^*_{ik2}}{\partial x_1}\right) + \frac{\frac{1}{2} a_{pq}}{\sqrt{\frac{-R}{2}}} \left\{ \frac{\partial(\varkappa Q^*_{kpq} + a_{kpq})}{\partial x_i} + \frac{\partial(\varkappa Q^*_{ipq} + a_{ipq})}{\partial x_k} \right\} -$$

$$- \frac{a_{pq}}{R} \left\{ (\varkappa Q^*_{kpq} + a_{kpq}) \frac{\partial \sqrt{\frac{-R}{2}}}{\partial x_i} + (\varkappa Q^*_{ipq} + a_{ipq}) \frac{\partial \sqrt{\frac{-R}{2}}}{\partial x_k} \right\}$$

also wegen (1) und (6)

$$S_{ik}^{(\varkappa)} = \varkappa \left\{ \frac{\partial a_{ik1}}{\partial x_2} - \frac{\partial a_{ik2}}{\partial x_1} + \tfrac{1}{2} \overset{*}{\alpha}_{\underline{pq}} \left(\frac{\partial \overset{*}{Q}_{kpq}}{\partial x_i} + \frac{\partial \overset{*}{Q}_{ipq}}{\partial x_k} \right) \right\} +$$
$$+ \left\{ \frac{\partial \overset{*}{Q}_{ik1}}{\partial x_2} - \frac{\partial \overset{*}{Q}_{ik2}}{\partial x_1} + \tfrac{1}{2} \overset{*}{\alpha}_{\underline{pq}} \left(\frac{\partial a_{kpq}}{\partial x_i} + \frac{\partial a_{ipq}}{\partial x_k} \right) \right\}$$

oder

$$S_{ik}^{(\varkappa)} = \varkappa\, n_{ik} + u_{ik} \qquad (\varkappa^2 \neq 1) \quad . \; . \; . \; . \; . \; . \; . \quad (11)$$

Für $\varkappa^2 = 1$ wird die Operation N sinnlos, da dann $\triangle_\varkappa$, $R_\varkappa$ und $Q_\varkappa$ gleich Null werden. Man findet jedoch durch Addition und Subtraktion von (8) und (9) zwei nicht verschwindende Kovarianten:

$$S_{ik}^{(1)} = \frac{\partial (a_{ik1} + \overset{*}{Q}_{ik1})}{\partial x_2} - \frac{\partial (a_{ik2} + \overset{*}{Q}_{ik2})}{\partial x_1} +$$
$$+ \tfrac{1}{2} \overset{*}{\alpha}_{\underline{pq}} \left\{ \frac{\partial (a_{kpq} + \overset{*}{Q}_{kpq})}{\partial x_i} + \frac{\partial (a_{ipq} + \overset{*}{Q}_{ipq})}{\partial x_k} \right\}$$

$$S_{ik}^{(-1)} = \frac{\partial (-a_{ik1} + \overset{*}{Q}_{ik1})}{\partial x_2} - \frac{\partial (-a_{ik2} + \overset{*}{Q}_{ik2})}{\partial x_1} -$$
$$- \tfrac{1}{2} \overset{*}{\alpha}_{\underline{pq}} \left\{ \frac{\partial (-a_{kpq} + \overset{*}{Q}_{kpq})}{\partial x_i} + \frac{\partial (-a_{ipq} + \overset{*}{Q}_{ipq})}{\partial x_k} \right\}.$$

Setzt man noch

$$f + Q^* = v_x^3 \qquad\qquad -f + Q^* = w_x^3$$

worin v_x^3 und w_x^3 reine Kuben sind [3]), so bekommen diese Kovarianten die neue Gestalt:

$$S_{ik}^{(1)} = \frac{\partial v_{ik1}}{\partial x_2} - \frac{\partial v_{ik2}}{\partial x_1} + \tfrac{1}{2} \overset{*}{\alpha}_{\underline{pq}} \left(\frac{\partial v_{kpq}}{\partial x_i} + \frac{\partial v_{ipq}}{\partial x_k} \right)$$

$$S_{ik}^{(-1)} = \frac{\partial w_{ik1}}{\partial x_2} - \frac{\partial w_{ik2}}{\partial x_1} - \tfrac{1}{2} \overset{*}{\alpha}_{\underline{pq}} \left(\frac{\partial w_{kpq}}{\partial x_i} + \frac{\partial w_{ipq}}{\partial x_k} \right)$$

Unterwirft man diese Kovarianten der Transformation, welche f überführt in die kanonische Gestalt (10), so wird

$$S^{(1)} = 2\, \frac{\partial a_{111}}{\partial x_2}\, dx^{1\,2} \qquad \text{und} \qquad S^{(-1)} = 2\, \frac{\partial a_{222}}{\partial x_1}\, dx^{2\,2}.$$

Ihre Diskriminanten sind Null.

$S_{ik}^{(1)}$ und $S_{ik}^{(-1)}$ sind also die entarteten Exemplare des Büschels $S_{ik}^{(\varkappa)} = \varkappa\, n_{ik} + u_{ik}$. Die Differentialkovarianten erster Ordnung der binären kubischen Differentialform führen zurück auf n_{ik} und ihre Ueberschiebung mit $\overset{*}{\alpha}_{ik}$.

Biochemistry. — *Behaviour of Microscopic Bodies consisting of Biocolloid Systems and suspended in an Aequeous Medium.* I. *Pulsating Vacuoles in Coacervate Drops.* By H. G. BUNGENBERG DE JONG. (Communicated by Prof. J. VAN DER HOEVE.)

(Communicated at the meeting of May 28, 1938.)

1. *General introduction.*

Experimental cytology studies the behaviour of microscopically small systems from biocolloids and it is faced by a remarkable difficulty in the interpretation of the morphological changes brought about under the influence of external or internal factors. It is tried to find some connection with the results of colloid-chemical researches, e.g. on sols and gels. However, the dimensions of the latter objects of research are usually such that their properties are practically determined only by those of the three-dimensional content of these systems and the influence of the bordering surfaces is not expressed in it.

With the objects of cytology, however, the proportion of bordering surface and content is totally different and the morphological changes observed by it consist of changes of the biocolloid systems, in which both, surface and content, are simultaneously contained and influence each other mutually.

For this reason we are inclined to think that, if colloid chemistry wishes to be of use to cytology, it will have to occupy itself with the study of microscopically small colloid bodies; by bodies is meant here: colloid systems surrounded by an external surface.

In the study of the coacervation phenomena the present writer already applied this method, in so far as not only the properties of the three-dimensional coacervates were examined but also those of microscopical coacervate drops. The significance of these systems in biology, which consequently may be regarded as fluid biocolloid bodies, has been stated elsewhere [1]).

In so far as the study of these fluid and other kinds of biocolloid bodies

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[1]) Summarizing articles on coacervation:
H. G. BUNGENBERG DE JONG, Die Koazervation und ihre Bedeutung für die Biologie, Protoplasma, **15**, 110 (1932).
H. G. BUNGENBERG DE JONG, La Coacervation et son importance en Biologie. Tome I et II. Hermann et Cie; Paris 1936.
H. G. BUNGENBERG DE JONG, Koazervation, Kolloid Z., **79**, 223, 334 (1937), **80**, 221, 350 (1937).

yields new results, which in cytology may be important as models, they will be briefly mentioned in the present and following communications.

2. *Coacervation of gum arabic sol with toluidine blue.*

If on an object glass at a small distance from each other are placed a drop of a concentrated toluidine blue solution and gum arabic solution (2 to 3 %) and both are covered by a cover glass, we observe under the microscope that in the mixing-zone of the two liquids coacervation of the gum arabic takes place. This coacervation belongs to the type of auto-complex coacervation and takes place with the negative gum arabic sol and many basic stains [1]).

By means of the above-mentioned method a zone of optimal coacervation is found, which is *not* situated where the colour is just no longer perceptible, but at some distance to the side of the stain solution. Here the largest coacervate drops are formed, which are opaque, owing to the stain they contain, while they lie in a liquid zone which is hardly stained.

This zone of optimal coacervation is explained by the fact that here gum arabic and stain are present in the most favourable mixing-ratio. On either side of this zone the coacervation decreases and the colour of the liquid containing the coacervate drops grows stronger. Towards the side of the stain solution the colour is typically that of an aequeous tiluidine blue solution, towards the side of the gum arabic solution the colour has shifted more to red ("meachromasia"). This shifting of the tint may be regarded as a result of the adsorption of the stain cation to the arabinate colloid anion. Since the ratio stain, gum arabic grows gradually smaller towards the side of the gum arabic solution, the coacervation soon stops here.

3. *Pulsating vacuoles.*

The coacervate drops sink on the object and stick to it, so that, when afterwards either of its own accord or by tilting or straining through with a filtering-paper the zone of contact of the gum arabic and stain solution is shifted, the coacervate drops will become surrounded by a medium with which they are no longer in equilibrium. They disappear at last, when they are surrounded by the gum arabic solution.

Fig. 1.

In the latter case, however, unexpected phenomena occur (cf. fig. 1). The coacervate drops, instead of being dissolved by exclusively peripheral

[1]) H. G. BUNGENBERG DE JONG and J. LENS, Biochem. Z., 254, 15 (1932).

action of the medium, besides absorb liquid from the medium, which is separated in the coacervate drops under formation of a vacuole.

The phenomena may be best observed when the zone of contact of the two liquids is slowly shifted. It is then seen that originally opaque coacervate drops increase in volume and in the centre grow transparent. They consist now of a still strongly stained, soon fairly thin spherical skin of coacervate enclosing a centrally situated vacuole which increases in volume. At a certain moment they burst and change into a small coacervate drop, which is again opaque. In its turn the latter can swell, forming a central vacuole, and burst again. As a rule there is not enough coacervate left then for the phenomenon to be repeated once more.

The phenomena taking place here are interesting from a biological point of view, since they remind us of the functioning of pulsating vacuoles. Meanwhile there is this difference that in the case of the coacervate drops the pulsating system is destroyed after a few pulsations. Nevertheless, a closer examination of the mechanism of these phenomena seems of importance to biology and in due time this will be discussed further. Incidentally it may be remarked that in principle the same phenomena were observed with some other basic stains, e.g. brilliant cresyl blue, but so far they were realized most beautifully with toluidine blue.

Summary.

1. The importance to cytology is discussed of the study of microscopical bodies consisting of biocolloid systems.

2. In coacervate drops, originated from gum arabic sol + toluidine blue, under certain conditions when they are no longer in equilibrium with their medium, pulsating vacuoles are formed.

3. Although this pulsating system is destroyed after a few pulsations, these phenomena may be of importance in biology.

May 1938.

Laboratory for Medical Chemistry at Leiden.

Biochemistry. — *Behaviour of Microscopic Bodies consisting of Biocolloid Systems and suspended in an Aequeous Medium.* II. *Formation of double-refractive "Membranes" on Gelatin Gel Globules by Tannin.* By H. G. BUNGENBERG DE JONG. (Communicated by Prof. J. VAN DER HOEVE.)

(Communicated at the meeting of May 28, 1938.)

1. *Introduction.*

On closer contact with the data of cytology the author thought it desirable to study more closely "biocolloid bodies", with the colloid system in the gel state.

The most simple case imaginable is that where only one biocolloid takes part in a gel. In the following a method is described to prepare microscopically small globules consisting of gelatin gel. The behaviour of these colloid bodies, in so far as they do not show strikingly new phenomena, will be discussed here only incidentally. We shall deal more elaborately with the action of tannin, where unexpected phenomena occur.

2. *Microscopically homogeneous gel globules from gelatin.*

a. *Preparation.*

The principle of preparation is this that at higher temperatures by means of a dehydrating substance a gelatin sol is caused to coacervate [1]), the still suspended coacervate globules are gelatinized by cooling, and the dehydrating substance is washed out by continued washing with distilled water.

According to the nature of the dehydrating substance and the circumstances under which the gelatinizing takes place, either microscopically homogeneous or heterogeneous (e.g. vacuolized) spherical colloid bodies may be formed.

Below the description is given of the preparation of homogeneous bodies by means of $(NH_4)_2SO_4$ and "Bacteriological gelatin" from the Glue and Gelatin Works "Delft" at Delft.

25 cc of 5 % gelatin solution is mixed at 50° with 50 cc of 20 % $(NH_4)_2SO_4$ solution in an Erlemeyer of 100 cc. By stirring with a thermometer the mixture may be kept at 50° for a short time, while

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[1]) For coacervation of gelatin sol with dehydrating substances, e.g. with Na_2SO_4, which case is analogous to the coacervation used here with $(NH_4)_2SO_4$, see more extensively: L. W. J. HOLLEMAN, H. G. BUNGENBERG DE JONG and R. S. TJADEN MODDERMAN, Kolloid Beih., **39**, 334 (1934).

H. G. BUNGENBERG DE JONG: BEHAVIOUR OF MICROSCOPIC BODIES CONSISTING OF BIOCOLLOID SYSTEMS AND SUSPENDED IN AN AEQUEOUS MEDIUM. II.

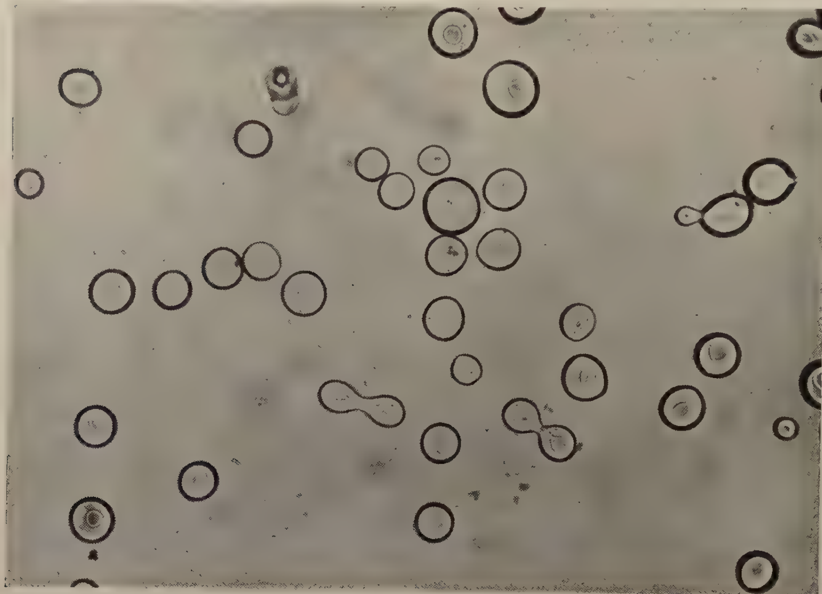

Fig. 1. (40 ×.)

Fig. 2. (40 ×.)

small coacervate drops unite to large ones. The coacervated mixture is then poured into a measure-glass with 250 cc of 10 % $(NH_4)_2SO_4$ at room-temperature (18-20°) and is then left to form a sediment (at least 1 hour). The liquid, which is still turbid with smaller fractions of gel globules, is then poured off to 15—20 cc and 250 cc of distilled water is added. This washing is repeated (at least 4 or 5 times) till $BaCl_2$ with the liquid no longer shows turbidity. This suspension may be kept at room-temperature during several weeks at least without decay, if put in a closed glass jar with some crystals of p-dichlorbenzol.

b. *Properties of the gel globules.*

The gel globules are microscopically homogeneous and between crossed nicols they show no indications of double refraction. They adhere strongly to the glass wall and consequently the preparation is accompanied by a considerable loss. The latter evil may be reduced by taking a paraffinated measure-glass for the sedimentation. The liquid, in which the gel globules after sufficient washing are suspended, yields a green colour with bromthymol blue, i.e. the PH is about 6.5.

They stain relatively faintly with highly dilute solutions of basic (crystal violet) as well as acid stains (Erythrosin). In an acid medium they stain strongly with Erythrosin and practically not with crystal violet, in an alkaline medium the opposite takes place. Further results concerning swelling, stainability, etc. will be published later.

3. *Action of tannin on the gelatin gel globules.*

a. *Formation of double-refrative "membranes".*

On placing a little of the gel globule suspension on an object glass and then 1 drop of dilute tannin solution (e.g. 1 %), we observe that on the periphery of the gel globules a strongly light-refracting marginal zone is formed, while the diameter of the gel globules decreases. Between crossed nicols the gel globules appear to become gradually double-refractive and, when the tannin has acted for some time, a highly luminous axial cross may be seen.

Figs 1 and 2 reproduce microphotographs of such and further advanced stages. Fig. 1 is made without nicols, fig. 2 with crossed nicols. In fig. 1 the strongly light-refracting nature of the periphery of the globules at the chosen position of the microscope becomes manifest as black rims.

At an equal position of the microscope the extent of the image visible with crossed nicols at first, therefore, is limited to the central part of the globules, situated within these black rims (apart from a fine line of light on the periphery of the globules visible at a suitable position of the microscope).

The double contoured circles which are visible in fig. 1, seemingly lying entirely centrally in the globules, are situated on the level of the

surface of contact gel globules/object glass and consequently represent the place where the marginal zone of the gel globules touches the glass. They are not found, therefore, in case of tannin acting on freely floating gel globules.

This morphological detail by the side of other indications (cf. c) makes us feel inclined to interpret the microscopical image as follows: Round a centrally situated gelatin gel, on which the tannin not yet had effect, there is peripherally a spherical skin, where this action has already taken place, which has become strongly double-refractive.

In a way we might speak, therefore, of the formation of a double-refractive "membrane" on the periphery of the gel globule [1]).

b. *Influence of acid and alkaline reaction (resorcin, etc.).*

The phenomena described above take place also in a slightly acid medium, likewise in a very weak alkaline medium. Besides, the tannin appears to adhere very strongly: The preparation which has become double-refractive may be washed for several minutes with slightly alkaline running tapwater without the double refraction being lost.

In a sufficiently alkaline medium, however, the described phenomena do not take place. All this points to the fact that tannin causes an analogous action here to that with the gelatin sol. There also dehydration takes place in an acid and neutral medium but in a more strongly alkaline medium it is prevented by formation of phenolate [2]).

Salts in not too high concentrations have not much effect either. If to the double-refractive preparation KCNS IN is added, the double refraction grows gradually weaker and the globules shrink. This may be connected with the neutralizing action of KCNS on the gel condition of the gelatin. Other substances which likewise neutralize the gelatinizing (phenols), such as resorcin, have the same effect.

Even with tannin (likewise a polyphenol) in a stronger concentration the same phenomena may be observed: With a 10 % solution at first the same phenomena are found as described in a). but gradually the double refraction diminishes and after contraction finally non-double-refractive coacervate drops are left.

c. *Transition of the gel globules into cup-shaped bodies.*

When tannin is left to act so long on the gelatin globules till the double contoured circles on the contact surface gel globules/object glass are easily perceptible and the preparation is then briefly washed with

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[1]) Further researches have to decide whether indeed still unaltered gelatin gel is to be found within the double-refractive marginal zone or that ultimately a vacuole is formed here.

[2]) For researches on the effect of tannin, crystalline, tannins and simpler phenols on sols of biocolloids, cf. H. G. BUNGENBERG DE JONG, Recueil trav. chim. d. Pays Bas, **42**, 437 (1923); **43**, 35 (1924); **46**, 727 (1927); **48**, 494 (1929).

running water and KCNS 1N is poured over it, they are more or less easily detached when the object glass is slightly moved backwards and forwards. Gradually the originally globular bodies show the transformations schematically reproduced in fig. 3, at the same time growing constantly smaller. These transformations are in agreement with the conception mentioned in a) that a peripheral double-refractive membrane

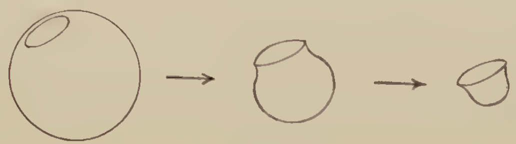

Fig. 3.

encloses an unaltered centrally situated gelatin gel. It should be borne in mind here that the KCNS acts on either of these parts. It will promote the swelling and dissolving of the central gelatin gel, so that the gel globules are now easily detached from the glass wall and the enclosed gel is dissolved, leaving only the membrane in the form of a spherical skin with a round opening.

However, the KCNS acts also on this "membrane", as has been described above in b) and this changes gradually into a highly viscous, no longer double-refractive coacervate, which of course tries to obtain a smaller surface. At the given initial form: spherical skin with round opening this smaller surface may be obtained simultaneously in two ways a) by diminution of the curvature radius of the spherical skin and b. by enlargement of the radius of the round opening. These two factors may explain the original transformations by which the cup-shaped bodies with a turned rim are formed. These transformations naturally are continued till finally the globular shape is reached.

d. *Stainability.*

The double-refractive globules stain very intensively with a dye (e.g. crystal violet), the double refraction still existing but usually no longer perceptible, owing to the very intensive staining. Moreover, the stain adheres strongly, since with only a slight weakening of the stain the preparation may be washed in running water and even may be left for hours in 1 % acetic acid.

e. *Behaviour of other substances with regard to the formation of double-refractive membranes.*

With a number of crystalline tannins: digalloyl glucose, chebulic acid and d. catechin the phenomena characteristic of tannin were not observed. Neither was this the case with resorcin which only produced a strong blackening from the periphery (formation of a large number of very small vacuoles).

Only with sodium pirate in an acid medium we also found the

formation of a double-refractive marginal zone. As compared to tannin, however, this was very weak and transient. Finally it can also be obtained with undiluted acetone; however, the strongly shrinking and consequently irregular globules neither show the beautiful image which we may obtain with tannin and become only slightly double-refractive.

f. *Discussion.*

From a theoretical point of view it is important that also other desolvating substances besides tannin may produce double refraction, which will be discussed more in detail in another publication.

It is further interesting that on heating of the double-refractive preparation the globules lose their double refraction and strongly shrink, forming coacervates. When the latter are cooled, a large amount of very small vacuoles are formed, which highly impede the optic investigation of double refraction. So far we did not succeed in observing double refraction in the few transparent spots of the coacervate at room-tempereture. It seems as if the manner in which the gelatin gel globules originated is of great importance to the possibility of formation of the double-refractive marginal zone with tannin. This may be expressed with a term from mineralogy by indicating these gelatin gel globules as "pseudomorphoses to gelatin coacervate drops". Apparently in the microscopical coacervate drops either the gelatin itself is already present in a more or less oriented condition or at the gelatinizing a certain orientation is formed from the surface of the coacervate drop to some distance inside. This orientation is still very imperfect but by suitable dehydating substances it may be improved so far that double refraction sets in.

In biology it is meanwhile important that tannin, which is so widely spread in the vegetable world, already in very small concentrations may contribute to the formation of strongly double-refractive membranes, since it occurs still clearly, though slowly, at a final concentration of 0.01 % tannin.

Summary.

1. The preparation of microscopical homogeneous gel globules of gelatin is described. The method followed is that of a pseudomorphosis to coacervate drops obtained with $(NH_4)_2SO_4$. The properties of these bodies are very briefly discussed.

2. A dilute tannin solution produces strongly double-refractive "membranes" on the surface of the globules mentioned in 1. Some properties of these formed double-refractive colloid bodies are described, the origin of the double refraction is discussed and the significance in biology is pointed out.

May 1938.

Laboratory for Medical Chemistry at Leiden.

Plantkunde. — *Snelle Bloei van de Narcis (N. Pseudonarcissus var. King Alfred)* I. Door ANNIE M. HARTSEMA en IDA LUYTEN. (Mededeeling No. 56 van het Laboratorium voor Plantenphysiologisch Onderzoek te Wageningen). (Communicated by Prof. A. H. BLAAUW).

(Communicated at the meeting of May 28, 1938.)

In ons onderzoek: temperatuur en strekkingsperiode van de Narcis I (Mededl. No. 35, 1932) is gebleken, dat het optimum voor de strekking gedurende de eerste 4 weken na het rooien eer bij 13° C dan bij 9° C ligt, maar dat 9° (het zg. *indirecte optimum*) *achterna* sneller bloei bewerkt. In 1932 werd hetzelfde nog eens vastgesteld. De bollen waren direkt na het rooien ontvangen op 28 Juli 1932 en werden gedurende 4 weken bij 7°, 9°, 13° en 20° C bewaard. In tabel 1 vindt men den invloed van deze temperaturen op de strekking van 1e scheede- en 1e loofblad en van de bloem, in vergelijking met de lengte van deze organen bij het begin van de proef. Ook nu weer bleek het *directe optimum* dus bij 13° te liggen.

TABEL 1. Gemiddelde lengte der organen in mm.

Fixeerdat.	Temp.	1e scheedeblad	1e loofblad	Bloem + stengel	Bloem
28 Juli '32	—	38.4	15.4	9.0	niet gemeten
24 Aug. '32	7°	42.6	18.4	10.8	6.8
Idem	9°	41.7	24.6	15.5	9.2
Idem	13°	**47.5**	**26.5**	**17.8**	**11.4**
Idem	20°	44.4	18.3	11.6	7.2

Nu werden op 24 Augustus van iedere behandeling telkens 16 bollen geplant en bij 9° geplaatst. Zoodra de gemiddelde neuslengte (buiten den bol) 3 cm bedroeg, werden de kistjes naar een kas van 17° C overgebracht, vervolgens bij een neuslengte van 6 cm naar een kas van 20°, terwijl bij het kleuren der eerste knoppen weer naar de kas van 17° teruggeplaatst werd. Bovendien werden groepjes uit 20° en uit 13° ook bij 7° geplant, evenals die uit 7°. In tabel 2 ziet men het resultaat van al deze groepen, zoowel wat het bereiken van de gewenschte neuslengte betreft, als den bloei. Het aantal dagen werd steeds berekend van het begin der proeven af.

Het is merkwaardig te zien dat bij het bereiken van 3 cm neuslengte de invloed van 13° nog het gunstigst lijkt. De neuslengte van 6 cm werd

TABEL 2.

Voorbehandeling	Plantdatum	Geplant bij:	Aant. dag. tot 3 cm	Aant. dag. tot 6 cm	A. dg. tot 1e bloem open	Datum 1e bloem open	Aant. dag. van 1e tot laatste bloem	Aantal bloemen
1932								
4 wk. 9°	24 Aug.	9°	114	**131**	**152**	**27 Dec.**	51	15 : 15
4 wk. 13°	"	9°	**107**	134	156	31 Dec.	69	15 : 15
4 wk. 20°	"	9°	109	142	172	16 Jan.	46	15 : 15
4 wk. 7°	24 Aug.	7°	127	137	160	4 Jan.	**20**	16 : 16
4 wk. 13°	"	7°	120	133	157	1 Jan.	27	16 : 16
4 wk. 20°	"	7°	125	143	159	3 Jan.	36	16 : 16

echter eerder bereikt na 4 wk. 9°; ook bij het in bloei komen was deze groep nog steeds de eerste (27 December).

Ook bij planten in 7° is een gunstige invloed van 4 wk. 13° nog merkbaar, al is het aantal dagen, noodig voor het bereiken van 3 cm neuslengte, hier 120 in plaats van 107 bij het planten in 9°. Bij 6 cm ziet men hoezeer 7° inhaalt, waardoor het begin van den bloei bij deze groep slechts 1 dag later is dan bij 9°. Dezelfde uitwerking van 7° zien we bij 4 wk. 20°; deze groep bloeit na planten in 7° zelfs 13 dagen eerder dan na planten in 9°.

De snelste bloei werd dus evenals in 1931 bereikt bij de met 9° voorbehandelde en in 9° geplante groep. In tegenstelling met 1931 duurde het ditmaal echter zeer lang eer alle bloemen van dezelfde groep open waren, nl. 51 dagen tegenover 13 dagen in 1931. Dit wordt waarschijnlijk ten deele veroorzaakt doordat nu vroeger geplant werd, zooals uit tabel 4 blijken zal. Des te opvallender is het dat bij planten in 7° de bloemen zooveel sneller na elkaar opengaan (zie de voorlaatste kolom). Wij zullen daarop nog vaker kunnen wijzen.

Nu moest ook worden nagegaan:

1e. of het overbrengen uit 9° naar hoogere temperaturen wellicht reeds bij het zichtbaar-worden der neuzen kon geschieden;

2e. of misschien andere dan de tot nu toe gekozen temperaturen gunstiger zouden zijn.

Daartoe werden de volgende combinaties gekozen: (tabel 3) waarbij steeds met 4 weken 13° voorbehandeld werd, terwijl alle groepen op 24 Augustus in 9° geplant werden. De snelste behandeling van tabel 2 (4 weken 9°) wordt hier nog eens herhaald. Deze blijkt ook in deze groepen den vroegsten bloei te geven. Daarop volgen de groepen die bij een neuslengte van 3 cm resp. naar 17°, 20° en 23° overgebracht werden, terwijl bij 6 cm al deze groepen weer in kas 20° kwamen. Het blijkt dat 17° het gunstigst is, zoowel voor het bereiken van 6 cm als voor den bloei; 23° was veel te hoog, de bloemen verdroogden alle. Na deze groepen volgen

TABEL 3.

Voorbehandeling 1932	Geplant bij	Aant. dag. tot zichtb. neuzen	over naar	Aant. dag. tot 3 cm	over naar	Aant. dag. tot 6 cm	over naar	Aant. dag. tot 1e bloem open	Datum 1e bloem open	Aant. dag. van 1e tot laatste bloem	Aant. bloemen
4 wek. 9°	9°	—	—	114	17°	131	20°	152	27 Dec.	51	15 : 15
4 wek. 13°	9°	—	—	107	17°	134	20°	156	31 Dec.	69	15 : 15
4 wek. 13°	9°	—	—	107	20°	140	20°	192	6 Feb.	28	5 : 15
4 wek. 13°	9°	—	—	107	23°	149	20°	—	—	—	0 : 16
4 wek. 13°	9°	74	17°	114	17°	149	20°	205	19 Feb.	—	2
4 wek. 13°	9°	74	20°	124	20°	174	20°	—	—	—	0
4 wek. 13°	9°	74	23°	131	23°	216	20°	—	—	—	0
4 wek. 13°	9°	—	—	104	17°	134	23°	157	1 Jan.	76	7 : 15

er 3, welke reeds met zichtbare neuzen naar 17°, 20° of 23° overgebracht werden. Van deze kwam alleen de eerste nog in bloei, maar het duurde zeer lang en er waren maar 2 bloemen. De beide andere groepen begonnen reeds te verdrogen voordat 6 cm bereikt was. Tenslotte werd nog een groep bij 6 cm naar 23° in plaats van naar 20° overgebracht: dit vertraagde den bloei enkele dagen, terwijl in 't geheel slechts 7 bloemen open kwamen!

Uit deze tabel blijkt wel heel duidelijk, dat de beste nabehandeling voorloopig bleef: *met neuslengte 3 cm naar 17°, 6 cm naar 20°*, terwijl hier zonder nader bewijs bij het kleuren der eerste bloemen teruggebracht wordt naar 17°.

In hetzelfde jaar werd ook nog de invloed van verschillende plantdatums nagegaan. Zoo werd direkt bij aankomst geplant, 4 weken later (zooals in de meeste proeven hierboven) en ook op 19 September, d.i. bijna 8 weken na aankomst. De laatste datum werd in overeenkomst met de proeven van 1931 gekozen (vergelijk Med. 35, blz. 807). In tabel 4 zijn de resultaten vermeld.

Heel duidelijk blijkt nu dat direkt planten geen voordeel biedt: de eerste bloem gaat pas 7 Jan. open, de andere volgen in zeer langzaam tempo, terwijl 4 geheel verdrogen. De tweede groep, die op 24 Augustus geplant werd, bloeide het eerst; de laat-geplante derde groep was enkele dagen later. Alle drie genoemde groepen bereikten ongeveer gelijktijdig 6 cm, hetgeen er op wijst dat de strekking in 9° sneller verloopt, naarmate men later plant. Toch kan men het planten niet willekeurig verschuiven, want uit de tabel blijkt wel dat in 1932 op 19 September de grens waarschijnlijk reeds bereikt was, omdat de bloei later begon dan

TABEL 4.

Voorbehandeling	Plantdatum	Geplant bij	Aant. dag. tot 3 cm	Aant. dag. tot 6 cm	Aant. dag. tot 1e bloem open	Datum 1e bloem open	Aant. dag. tusschen 1e en laatste bloem	Aantal bloemen
	1932							
—	28 Juli	9°	107	130	163	7 Jan.	72	12 : 16
28 dagen 9°	24 Aug.	9°	115	132	153	27 Dec.	51	15 : 15
54 dagen 9°	19 Sept.	9°	124	134	156	30 Dec.	24	15 : 16
	1931							
51 dagen 9°	18 Sept.	9°	123	129	146	22 Dec.	13	20 : 20

in 1931 na 51 dagen 9°. Tevens blijkt uit deze tabel dat laat planten een goeden invloed heeft op het snel na elkaar in bloei komen van de planten van één proef.

In 1933 werd nog eens weer zoowel met 9° als met 7° behandeld en bij dezelfde temperatuur geplant. Als plantdatum werd 1 September gekozen, behalve voor 1 groep, die pas 20 September geplant werd. De bollen werden reeds 22 Juli ontvangen. In afwijking met de vorige jaren werden nu slechts 6 bollen per kistje geplant, iedere groep bestond echter uit 3 kistjes, d.i. 18 bollen.

TABEL 5.

Voorbehandeling	Plantdatum 1933	Geplant bij	Aant. dag. tot 3 cm	Aant. dag. tot 6 cm	Aant. dag. tot 1e bloem open	Datum 1e bloem open	Aant. dag. tusschen 1e en laatste bloem	Aantal bloemen
41 dagen 7°	1 Sept.	7°	114	128	155	24 Dec.	22	18 : 18
41 dagen 7°	1 Sept.	9°	112	133	157	26 Dec.	42	17 : 17
41 dagen 9°	1 Sept.	9°	108	130	168	6 Jan.	42	18 : 18
60 dagen 9°	20 Sept.	9°	122	135	157	26 Dec.	23	18 : 18

Ditmaal is de met 7° behandelde groep het eerst in bloei, hoewel de neuslengte van 3 cm later bereikt werd dan bij de beide volgende groepen. Daarop volgen de 2e en de 4e groep, die op 26 Dec. in bloei komen, terwijl het laatst (op 6 Jan.) de met 9° voorbehandelde en op 1 September in 9° geplante groep in bloei komt. Met 7° voorbehandelde bollen bloeiden bij planten in 9° iets later dan bij planten in 7°. Overigens is de bloei van alle groepen goed; de snelheid van het in bloei komen verschilt nogal, is het best bij de eerste en vierde groep. Hierdoor worden

dus de ervaringen van 1932 bevestigd, dat *zoowel planten bij 7° als 3 weken later planten* bij 9° *gunstig werkt op het snel opengaan van alle bloemen.* Bij iedere groep kwam één bol voor die pas veel later bloeide, deze bollen zijn echter niet meegeteld bij het vaststellen van het aantal dagen voor de voorlaatste kolom. Ook een voorlooper uit de 1e groep, die 8 dagen eerder bloeide, werd niet meegerekend.

Omdat wij vermoedden dat de aangenomen neuslengten, resp. 3 en 6 cm buiten den bol, misschien niet de gunstigste waren, deden wij in 1934 enkele proeven om dit na te gaan. Wij hoopten ook een gemakkelijker maatstaf te vinden in het uitgroeien van de loofbladen buiten de scheede. Inderdaad bleek bij een neuslengte van ± 4 cm het eerste loofblad juist zichtbaar, terwijl het bij een neuslengte van ± 7 cm ongeveer 3 cm buiten de scheede stak. Om de proeven onderling te kunnen vergelijken, bleven wij echter de neuslengte buiten den bol meten.

Verder werd een deel der proeven bij 3 (resp. 4) cm naar 13° overgebracht in plaats van naar 17° zooals tot nu toe gebruikelijk was. Bij 6 resp. 7 cm werden alle groepen naar 20° gebracht. In de volgende tabel 6 vindt men het resultaat van deze bij verschillende neuslengten overgebrachte groepen. Alle bollen waren op 2 Augustus ontvangen en werden op 1 September geplant. Om het aantal proeven niet te veel uit te breiden werd de invloed van 3 weken later planten ditmaal niet nagegaan. Iedere groep bestond uit 4 kistjes of 24 bollen.

Tabel 6.

Voorbehandeling	Plantdatum 1934	Geplant bij	Aant. dag. tot 3 cm	Over naar	Aant. dag. tot 6 cm	Over naar	Aant. dag. tot 1e bloem open	Datum 1e bloem open	Aant. dag. tusschen 1e en laatste bloem	Aantal bloemen
30 dagen 7°	1 Sept.	7°	110	17°	119	20°	139	19 Dec.	15	23 : 24
30 dagen 7°	1 Sept.	7°	110	13°	120	20°	138	18 Dec.	9	24 : 24
30 dagen 9°	1 Sept.	9°	107	17°	119	20°	138	18 Dec.	25	24 : 24
30 dagen 9°	1 Sept.	9°	107	13°	123	20°	141	21 Dec.	18	24 : 24
			4 cm		7 cm					
30 dagen 7°	1 Sept.	7°	123	17°	127	20°	144	24 Dec.	14	24 : 24
30 dagen 7°	1 Sept.	7°	123	13°	130	20°	146	26 Dec.	12	23 : 24
30 dagen 9°	1 Sept.	9°	121	17°	125	20°	143	23 Dec.	16	24 : 24
30 dagen 9°	1 Sept.	9°	123	13°	128	20°	146	26 Dec.	12	22 : 24

Het blijkt wel uit tabel 6 dat het, met het oog op het begin van den bloei, minder gunstig is de bollen in de lage temperatuur te laten totdat

de neuslengte 4 cm bedraagt in plaats van 3 cm: dit begin wordt er ongeveer 5 dagen door vertraagd. Maar het is opvallend dat de verdere strekking zoo vlot verloopt als men wacht met het overgaan naar hoogere temperatuur totdat de neuslengte 4 cm bedraagt. Dit uit zich bij 9° ook in de vermindering van het aantal dagen tusschen het verschijnen van de eerste en de laatste bloem. Maar in dit jaar was dat aantal dagen ook bij het overbrengen bij 3 cm belangrijk minder dan in vorige jaren. De bij 7° behandelde en geplante groepen vertoonen geen verschil in de snelheid van het opengaan van alle bloemen, of men reeds bij 3 cm overbrengt of daarmee wacht tot 4 cm. Daarom geven wij bij deze behandeling met 7° dus *de voorkeur aan overbrengen bij 3 cm neuslengte* uit den bol, waardoor de bloei 5 à 6 dagen eerder begint.

Letten wij alleen op het begin van den bloei, dan is er dit jaar niet veel verschil tusschen 7° en 9°: brachten wij de bollen, zooals tot nu toe de gewoonte was, bij 3 cm naar 17°, zoo bleek 9° 1 dag eerder in bloei te komen dan 7°. Werd echter naar 13° overgebracht, dan was 7° 3 dagen eerder dan 9°. Overbrengen bij een neuslengte van 3 cm naar 13° in plaats van naar 17° bleek dus alléén gunstig bij de in 7° geplante groep; dezelfde groep bij 4 cm neuslengte overgebracht naar 13° begon 2 dagen later te bloeien dan bij overbrengen naar 17°. Ook de met 9° behandelde groepen vertoonden enkele dagen verlating van den bloei door het overbrengen naar 13° in plaats van naar 17°, zoodat *in het vervolg steeds 17° toegepast werd.*

Het voortzetten van de koude-behandeling totdat 4 cm bereikt is, biedt dus alleen bij 9° voordeel. Om onze proeven onderling te kunnen vergelijken, hebben wij in het vervolg zoowel na 7° als 9° steeds bij 4 cm neuslengte overgebracht naar 17° en bij 7 cm naar 20°, terwijl steeds bij het opengaan der eerste bloemen naar 17° teruggebracht werd. In 1935 werd slechts één behandeling toegepast, nl. met 7°, terwijl in 1936 nog eens 7° en 9° vergeleken werden. Behalve op 1 September, werd nu ook weer direkt na de ontvangst der bollen geplant om na te gaan, welk effekt dit zou geven, nu wij de lage-temperatuur-behandeling pas bij 4 cm

TABEL 7.

Voorbehandeling	Plantdatum	Geplant bij	Aant. dag. tot 4 cm	Aant. dag. tot 7 cm	Aant. dag. tot 1e bloem open	Datum 1e bloem open	Aant. dag. tusschen 1e en laatste bloem	Aantal bloemen
24 dagen 7°	1 Sept. '35	7°	119	125	140	26 Dec. '35	10	17 : 18
24 dagen 7°	1 Sept. '36	7°	115	121	137	23 Dec. '36	9	18 : 18
—	8 Aug. '36	7°	115	124	139	25 Dec. '36	10	18 : 18
24 dagen 9°	1 Sept. '36	9°	115	124	140	26 Dec. '36	9	18 : 18
—	8 Aug. '36	9°	115	124	139	25 Dec. '36	10	18 : 18

ANNIE M. HARTSEMA EN IDA LUYTEN: SNELLE BLOEI VAN DE NARCIS (N. PSEUDONARCISSUS VAR. KING ALFRED).

PLAAT 1

Narcis King Alfred 9°, 21 Sept. geplant. Begin van de bloei 21 December, foto 27 Dec. 1937 ($5\frac{1}{2}\times$ verkleind).

neuslengte stop zetten. In tabel 7 zijn de resultaten van beide jaren vereenigd.

Voor iedere proef werden 18 bollen genomen. De data van het in bloei komen verschillen zeer weinig; 7° op 1 September geplant is de vroegste van alle groepen. In tegenstelling met onze ervaring in 1932 (zie tabel 4) bloeien de vroeg geplante groepen tegelijk met de andere groepen en even goed. In al deze proeven gaan de bloemen zeer snel na elkaar open, sneller nog dan in 1934.

Tenslotte werden in 1937 nog eens 7° en 9° vergeleken bij direkt, na ± 5 weken (op 1 September) en na ± 8 weken planten. De resultaten daarvan zijn in tabel 8 te vinden. Het aantal dagen wordt steeds na het begin der proeven berekend.

Wat het bereiken van de verschillende neuslengten en het begin van

TABEL 8.

Voorbehandeling	Plantdatum 1937	Geplant bij	Aant. dag. tot 4 cm	Aant. dag. tot 7 cm	Aant. dag. tot 1e bloem open	Datum 1e bloem open	Aant. dag. tusschen 1e en laatste bloem	Aantal bloemen
—	27 Juli	7°	118	125	143	17 Dec.	13	18 : 18
36 dagen 7°	1 Sept.	7°	123	127	143	17 Dec.	11	18 : 18
56 dagen 7°	21 Sept.	7°	125	129	147	21 Dec.	7	18 : 18
—	27 Juli	9°	118	125	145	19 Dec.	23	18 : 18
36 dagen 9°	1 Sept.	9°	121	126	143	17 Dec.	15	18 : 18
56 dagen 9°	21 Sept.	9°	125	129	147	21 Dec.	9	18 : 18

den bloei betreft, vinden wij ditmaal al heel weinig verschil tusschen de bij 9° of bij 7° behandelde groepen die op denzelfden datum geplant zijn. Alleen bij de direkt geplante groepen begon de bloei bij 9° 2 dagen later dan bij 7°. De 't laatst geplante groepen bloeien 4 dagen later dan alle andere. De invloed van den tijd van planten uit zich vooral in de snelheid waarmee alle bloemen van een proef opengaan. Ook bij de met 7° behandelde groepen, maar vooral bij de met 9° behandelde blijkt duidelijk het verschil tusschen direct planten en later planten. Plaat 1 geeft een beeld van de op 21 September geplante, met 9° behandelde groep op 27 December, d.i. 6 dagen nadat de eerste bloem open ging.

Op grond van deze en de vorige ervaringen moeten wij dus direct (± 1 Aug.) planten ontraden, omdat de vlotheid, waarmee de bloemen van één groep opengaan er ongunstig door beïnvloed wordt. Laat planten (± 20 Sept.) geeft enkele dagen verlating van het begin van den bloei, maar dit wordt vergoed door het snel in bloei komen van de geheele groep. VAN SLOGTEREN ontraadt zeer laat (± half October) planten,

omdat dit een slechten invloed heeft op de beworteling. (Meded. Laborat. v. Bloembollenonderzoek No. 47, 1933).

BEYER en VAN SLOGTEREN (Meded. Laborat. v. Bloembollenonderzoek No. 45, 47 en 49) konden zeer veel variëteiten van Narcissen vóór Kerstmis in bloei trekken en zij zijn daarbij vrijwel tot dezelfde behandeling gekomen, als door ons gevonden werd. Een diepgaande vergelijking is echter niet mogelijk, daar zij op dezelfde data proeven met verschillende spruitlengten naar de kas overbrengen, terwijl wij de proeven direct in de kas brengen wanneer een bepaalde spruitlengte bereikt is. Zij wijzen ook zelf op het belang van het juiste tijdstip van overbrengen (Meded. 45, blz. 26). Nadere gegevens over het verschil in openkomen van de eerste en de laatste bloem, dus over de vlotheid van het in bloei komen van een groep, worden niet vermeld. Dit bleek in onze proeven met King Alfred toch wel een punt van belang te zijn.

In de onlangs verschenen mededeeling No. 5 van den proefschooltuin te Lisse maakt VOLKERSZ de resultaten bekend van zijn proeven met verschillende narcis-variëteiten, die een vervolg zijn van het onderzoek vastgelegd in mededeeling No. 4. In 1937—'38 bracht hij o.a. ook King Alfred vroeg in bloei. Hij kon aantoonen, dat de bloei ook van deze variëteit vervroegd werd door een behandeling met 34° gedurende 4 dagen direkt na het rooien. Het is jammer, dat daarin de data van het in bloei komen van King Alfred niet genoemd worden, juist omdat wij vonden dat dit in 1937 vroeger was dan in andere jaren. Overigens komt ook VOLKERSZ tot vrijwel dezelfde voorschriften: na korte voorbehandeling bij hoogere temperaturen brengt hij de bollen bij 8 à 9° C, plant ze ± 1 October en brengt ze naar een matig verwarmde kas (17 à 18° C) bij een spruitlengte van ± 6 cm.

In de volgende tabel 9 geven wij nog eens een overzicht van de met 9° en 7° behaalde resultaten in de verschillende jaren bij planten op, of omstreeks 1 September.

In 1932 was een week vroeger geplant nl. op 24 Augustus; bij het vergelijken met andere jaren moeten wij dit in acht nemen.

Vergelijken we de data van het in bloei komen bij 7° en 9°, dan zien we dat alleen in 1932 (toen op 24 Augustus geplant was) 9° eerder was dan 7°. In 1933 was 7° vroeger dan 9° en in de volgende jaren was er geen verschil, behalve in 1936 toen 7° weer enkele dagen vroeger was. *Letten we echter op het aantal dagen dat verloopt tusschen het opengaan van de eerste en de laatste bloemen dan blijkt 7° steeds gunstiger te zijn dan 9°*, vooral zoolang reeds bij een neuslengte van 3 cm buiten den bol overgebracht werd naar 17°.

Letten we op den *trektijd*, d.i. het aantal dagen, gerekend van het begin der proeven tot het begin van den bloei, dan blijkt deze behalve in de jaren 1932 en 1933 vrij constant te zijn. Men bedenke dat om de cijfers van 1932-1934 (overbrenging 3 en 6 cm) en 1934-1937 (overbrenging 4 en 7 cm) te kunnen vergelijken bij den trektijd van 1932-1934 5 dagen

TABEL 9. Overzicht van de resultaten van behandeling met 9° en 7° in de jaren 1932—1937.

Jaar	Plant-datum	Duur van de voor-behande-ling	9° Aant. dag. tot 1e bloem open	9° Datum 1e bloem open	9° Aant. dag. 1e tot laatste bloem	7° Aant. dag. tot 1e bloem open	7° Datum 1e bloem open	7° Aant. dag. 1e tot laatste bloem
Overgebracht bij 3 cm naar 17°, 6 cm naar 20°.								
1932	24 Aug.	28 dagen	153	27 Dec.	51	161	4 Jan.	20
1933	1 Sept.	41 dagen	168	6 Jan.	42	155	24 Dec.	22
1934	1 Sept.	30 dagen	138	18 Dec.	25	139	19 Dec.	15
Overgebracht bij 4 cm naar 17°, 7 cm naar 20°.								
1934	1 Sept.	30 dagen	143	23 Dec.	16	144	24 Dec.	14
1935	1 Sept.	24 dagen	—	—	—	140	26 Dec.	10
1936	1 Sept.	24 dagen	140	26 Dec.	9	137	23 Dec.	9
1937	1 Sept.	36 dagen	137	17 Dec.	15	143	17 Dec.	11

opgeteld moeten worden (zie blz. 656). De trektijd wordt dus voor 9° resp. 158, 173, 143, 143, 140, 137 dagen. Men vraagt zich af wat de oorzaak van den langen trektijd van 1932 en '33 kan zijn. Immers de behandeling is geheel dezelfde. Het ligt voor de hand, te zoeken naar mogelijke verschillen in ontwikkeling reeds bij het rooien. We merken hierbij op, dat het tijdstip van rooien in de verschillende jaren uiteenloopt, daar dit met het afsterven der loofbladen, afhankelijk van de weersgesteldheid, samenhangt. De bollen werden ons steeds direct na het rooien toegezonden.

TABEL 10. Gemiddelde lengten der verschillende organen in mm en het stadium van de bloemen.

Fixeerdatum	1e loofblad	Bloem-stengel	Bloem	Stadium
28 Juli 1932	15.4	9.0	—	IX (4×) VIII+ tot VIII—IX (6×)
22 Juli 1933	16.1	9.3	5.9	IX (3×) VIII+ tot IX⁻ (7×)
3 Aug. 1934	18.2	9.7	6.0	IX (6×) VIII++ tot IX⁻ (4×)
9 Aug. 1935	13.7	8.5	5.2	IX (9×) VIII—IX (1×)
8 Aug. 1936	25.8	16.5	10.1	IX (10×)
27 Juli 1937	16.1	9.9	6.0	IX (9×) VIII++ (1×)

Uit tabel 10, waarin wij een overzicht der gemiddelde lengten van de belangrijkste organen bij 10 bollen (gefixeerd bij het begin der proeven)

geven, blijkt dat de lengten der organen vrijwel gelijk zijn op het tijdstip van rooien. Een uitzondering vormt 1936; de verschillende organen zijn hier duidelijk grooter dan bijv. in 1935, toen ongeveer op denzelfden datum gerooid werd. Toch zien we uit tabel 9, dat de bloei in 1936 slechts enkele dagen eerder begon. De bloemen zijn, zooals te verwachten was (HUISMAN en HARTSEMA), bijna geheel gevormd, alleen de bijkroon moest bij een deel der bloemen (stad. VIII+ tot IX) nog afgemaakt worden. In de ontwikkeling op het oogenblik van rooien hebben wij bij de narcis dus geen maatstaf voor het meer of minder vroeg bloeien. De lange trektijd voor 1932 en 1933 kunnen wij dus hieruit niet aflezen. Dat de weersgesteldheid van de laatste weken voor het rooien mogelijk hierop van invloed kan zijn, zullen wij aan het einde van het 2de gedeelte nader bespreken. Daar vindt men ook onze conclusies voor de practijk.

Wageningen, April 1938.

A summary will be given at the end of the second part in the next number.

Botany. — *Protoplasmic streaming in relation to spiral growth of Phycomyces.* By L. J. Jos. Pop. (Communicated by Prof. L. G. M. Baas Becking).

(Communicated at the meeting of May 28, 1938.)

After the publication by Oort: "The spiral growth of Phycomyces" protoplasmic streaming was measured by me in Febr. 1932 both in young and in old sporangiophores. This was done because a connection might be expected between the direction of the protoplasmic streaming and the spiral growth of the cell wall, according to a hypothesis of H. J. Denham in extension of the work of Crüger and Dippel (cf. van Iterson, 1927).

Material and method.

Phycomyces Blakesleeanus was cultivated in another way as was done by Blaauw, de Boer, Oort, a.o., viz. on malt-agar in glass-boxes, 6 cm high (cf. Burgeff). On this sterilized culture medium (500 cc malt extract + 1000 cc aq. dest. + 30 gr. agar) Phycomyces, both the "+" and "—" strain grew equally well. After sufficient growth of the sporangiophores had taken place cubes of agar (± ½ cc) with only one sporangiophore were cut out from the culture medium, the cube was cut off obliquely so that the sporangiophore assumed a horizontal position on the object-glass. According to Trzebinski the lesion by cutting through the mycelium with a Gillette blade at a distance of a few millimetres from the sporangiophore does not greatly affect the activity of Phycomyces, as the injured spot closes immediately. The sporangiophores remained turgid, as might be expected. The abnormal horizontal position of the sporangiophore could be avoided similarly as was done by Oort by placing the sporangiophores during a part of the growth horizontally, but due to geotropical response a double curve in a sporangiophore ensues which creates an undesirable condition for measuring the protoplasmic streaming. It would be better to measure the protoplasmic streaming with a horizontal microscope, the sporangiophore remaining vertical, afterwards turning the microscope into a vertical position. In this way the influence of gravity upon protoplasmic streaming might be tested.

Results and preliminary discussion.

Some hundred sporangiophores (with and without sporangia) were measured at that time and in most cases (with an enlargement of ± 1600 ×

under darkfield illumination) streaming of the protoplasm in a spiral could not be found (a few cases excepted, just before the stopping of the streaming or locally in a few sporangiophores). As a matter of fact the streaming proved to be nearly always parallel to the long axis of the sporangiophore.

Another phenomenon that was found at that time, was the peculiar way the protoplasm streams. After my investigation in 1932 and again by my experiments during the summer of last year I came to the conclusion that the protoplasm streams in two concentric tubes. This result is in agreement with the work of KIRCHHEIMER (1933), although I cannot agree with KIRCHHEIMER as to the directions of the streaming: this discrepancy may be due to the inversion of the image in the microscope.

According to my opinion there exists a tubular ascent of the protoplasm around a central sap-vacuole towards the top of the sporangiophore, where the protoplasm turns back in a tubular stream towards the substratum between the ascending stream and the outer wall (cf. fig. 1). The velocity, moreover, of the stream towards the substratum was usually greater than that of the stream towards the top, as will appear from

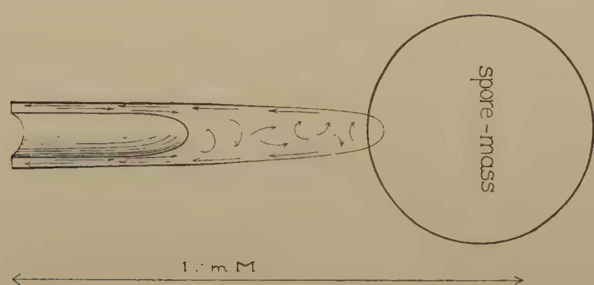

Fig. 1.

table 1, where some average velocities are given, calculated from at least five velocities measured by the speed of microsomes of about the same diameter.

Indications of the streaming in two opposite directions, as mentioned above, may also be derived from some quotations of other investigators, cf. e.g. KLEIN (1872), who writes of protoplasmic streaming in Pilobolus: "Sie fliessen in vielen Strömchen nicht bloss nach einer, sondern nach entgegengesetzten Richtungen". "Die Richtung ist vorwiegend nach oben und sich gleichbleibend; nur selten finden sich zwischen den vielen aufwärtssteigenden Strömchen auch zurückkehrende, aber einzelne sind immer vorhanden". "Sie geht selbst unter Deckglas oft Stunden lang ungestört und ungeändert vor sich", or VAN TIEGHEM (1875), who states in the case of Phycomyces nitens; "Après la maturité du sporange, ces derniers se vident progressivement, mais le protoplasme pariétal y persiste assez longtemps sous forme d'une couche très-mince, marquée de stries parallèles, verticales ou inclinées en hélice. Les granules montent le long

TABLE I.

	First layer of protoplasm (nearest to cover-glass and near the wall) Streaming from top towards substratum	Second layer of protoplasm (near the vacuole) Streaming from substratum towards top	Third layer of protoplasm (near the vacuole) Streaming towards top	Fourth layer of protoplasm (nearest to Object-glass and near the wall) Streaming towards substratum
Old sporangiophore measured $1\frac{1}{2}$ cm from top	3.4 μ/sec.	2.4 μ/sec.	2.1 μ/sec.	3.5 μ/sec.
Younger sporangiophore measured $1\frac{1}{2}$ cm from top	4.3 μ/sec.	2.8 μ/sec.	3.2 μ/sec.	4.5 μ/sec.
Young sporangiophore measured 1.3 cm from top	3.9 μ/sec.	2.6 μ/sec.	2.3 μ/sec.	3.3 μ/sec.

de certaines autres stries qui alternent quelquefois assez régulièrement avec les premières.", while OORT—ROELOFSEN (1932) also states; "Hauptsächlich bewegen Sie (z.w. die Plasmaströmchen) sich der Spitze hin, aber fast immer findet man auch wenige Fädchen, die basalwärts gehen". There are, moreover, apparently more microsomes in the ascending than in the descending stream. There is perhaps some material used in the growing region for building up the cell wall. Another possibility is the greater diameter of the descending stream that causes this apparent difference, the more so as in the ascending stream the currents of microsomes are more locally, while in the outer descending stream the microsomes are more equally divided over the whole diameter. Both the opinions of BENECKE—JOST (1923, p. 430); "Das beweist nun, dass nicht etwa der Zellsaft, sondern die peripheren Plasmamassen die Rolle, die das Substrat bei den Amöben spielt, übernehmen," and that of SIERP in "STRASBURGER's Lehrbuch der Botanik" (1931, p. 272); "Für das Zustandekommen dieser Bewegungen kommen wohl Aenderungen in der Oberflächenspannungen zwischen Protoplasma und Zellsaft im Frage," might both partially apply at the same time for Phycomyces, for the currents of microsomes in the two directions are (as seen under the microscope) not always next to each other, but certainly often enough beneath each other, so that the descending stream certainly does not come in contact with the whole surface of the cell sap, while a contact between the ascending stream and the peripheral layers of the descending stream never exists.

The opinion of CASTLE and of OORT—ROELOFSEN, where these investigators state about the growth-zone; "Die Strömung ist gering und ohne bevorzugte Richtung" cannot be considered as generally valid. The

growth-zone is, according to ERRERA and OORT limited to the region of ½—2 mm below the point of attachment of the spore-mass. From OORT and ROELOFSEN's opinion conclusions may be derived against the directing effect of protoplasmic streaming in the formation of the tubular structure of the primary cell wall. Against this opinion, however, a directed streaming up towards 100 μ [1]) below the sporangium was found more than once, although the Brownian movement of the greater number of microsomes at the top of the sporangiophore makes clear observation difficult.

Other culture-methods.

In August 1937 during a short working period at the Botanical Laboratory of Leyden the observations were repeated, as it was imaginable that the difference in culture method used by me in contrast with OORT or CASTLE might be responsible for the difference in direction of the protoplasmic streaming. For this reason the "+" and "—" strains were cultured not only on sterilized malt-agar, as was done before, but also on moist bread, as was done by OORT. Furthermore other culture media were also used in relation to the work of LINDNER, viz. sterilized 2 % glucose-agar, 2 % fructose-agar, 2 % amylum solubile-agar and 3 % lindseed-agar, which was used also afterwards by DE BOER in his investigation on the metabolism of Phycomyces.

Experiments and discussion.

The cultures were placed in the dark at 19° C and at 26° C. The growth varies a good deal. At 26° C almost no growth occurred. At 19° C, however, after five days the growth, both of the "+" and of the "—" strain, was abundant on malt-agar and moist bread (the sporangia were 6 cm or longer). On linseed-agar the sporangiophores have a different appearance. They are only 2 cm long, very thick, blueish-yellow and without sporangia, or at the utmost with very small and black sporangia. On amylum-agar only a very few sporangiophores are present, which were rather thin, while on glucose and fructose-agar some mycelium was present and no sporangia ever developed.

In a second series the same results were obtained except on glucose- and fructose-agar, where at this time after five days very thin sporangiophores were present, on glucose they were without sporangia and on fructose they showed very small sporangia. No difference in growth could be stated between the "+" and "—" strain on all the culture media, which fact also applies to the type of protoplasmic streaming of both the "+" and "—" strain, as was evident from my later observations on protoplasmic

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1) CASTLE (1936) states in his "Origin of spiral growth in Phycomyces, p. 498; "Another intrinsic difficulty is that the growth zone does not begin immediately below the sporangium but at a variable distance of 0.1 to 0.2 mm".

streaming of Phycomyces, where sporangiophores of both strains were used, often grown on different media.

During this time the protoplasmic streaming was observed with a binocular Zeiss microscope and measured nearly always with a "Kardioid Dunkelfeld" condenser and an enlargement of 630 ×. The results in general were the same, also with the sporangiophores which were cultivated on moist bread, as was stated in my earlier observation. The only difference was that the measured velocities (3 μ/sec. towards the substrate and 2 μ/sec. towards the top of the sporangiophore) were this time smaller than in Febr. 1932, but this can easily be caused by the greater distance from the spore-mass at which the streaming was measured in 1932. The velocity of protoplasmic streaming decreases when measuring more towards the spore-mass of the sporangiophore as is evident from the following table, where the times in seconds are given (an average from at least five measurements), required by microsomes to cover a distance of 29 μ at different distances from the top (the data are given in chronological sequence).

TABLE II.

Sporangiophore "—" strain, on linseed-agar Length 1.6 cm	Measured at a distance from the top of:	Velocity of the streaming over 29 μ in seconds:	
		towards the top:	towards the substrate:
	7 mm	9.7	8.2
	4.5 „	12.6	9.6
	1.5 „	15.5	12.5
	7 „	13.8	9.5
	4 „	15.5	12.7
	1 „	no streaming	no streaming
	7 „	14.4	11.4
	4 „	no streaming	14.1
	1.5 „	no streaming	no streaming

Where three times after another the streaming velocities were measured at three different distances from the spore-mass and the velocity at 7 mm from the top was always greater than the velocity, which was observed before at 4.5 mm or 1.5 mm from the top, it was evident that the differences in velocity at various distances from the top could not be caused by the slow decrease of the protoplasmic stream in the whole sporangiophore, which decrease is also evident from the above table.

From this experiment it also appears (and it was confirmed in ± 20 observations) that the streaming towards the top ceases earlier than the

streaming towards the substrate of the sporangiophores. The differences of the streaming-velocity in proportion to the distance from the top (where the velocity was measured) and the discrepancy in velocity between the ascending and the descending stream could at first view be ascribed to the influence of gravity. Though this is a factor which plays probably a part in the streaming velocity in the normal vertical position of the sporangiophores, it does not seem to hold in these measurements as all these velocities were measured in a horizontal position of the sporangiophore. In my opinion there is a better chance that these differences are caused by the thickness of the protoplasmic layers, which thickness is greater at the top where no cell sap is present and gradually diminishes in the sporangiophore towards the substratum. The objection could be made that the decrease of the protoplasmic streaming, close to the spore-mass, was a local phenomenon in an individual ridge of microsomes. This cannot, however, be accepted as an explanation as will be apparent from several sets of observations, one of which is shown in table 3. In this special case it was possible, in spite of the enlargement used (630 ×), to measure at the same time the protoplasmic stream above- and below the vacuole, because the sporangiophore was only 60 μ in diameter. Almost the same velocities were observed in the four currents, also at various distances from the spore-mass.

TABLE III.

Sporangiophore, length 6 cm, thickness 60 μ
Streaming velocity measured over a distance of 29 μ

At a distance from the spore-mass of:	Time in seconds:			
8 mm	↓ 9 → ↓ 8.6 → ↓ 9.6	↑ 11.2 ↑ 12.4 ↑ 13.2	↑ 11.8 ↑ 11.8 ↑ 12.8	↓ 9.8 ↓ 9.2 ↓ 9.8 ←
5 mm	↓ 13.4 → ↓ 13.6	↑ 16.4 ↑ 16.6	↑ 16.6 ↑ 16.8	↓ 13 2 ↓ 13.6 ←

In table 3, where the times necessary in covering a distance of 29 μ are shown, the vertical arrows refer to the direction of the protoplasmic stream, viz. towards the top of the sporangiophore ↑ or towards the substrate ↓, while the horizontal arrows refer to the succession in time of the recorded measurements. When the arrow points to the right the stream, which approximates most the objective of the microscope, was first measured and the contrary (viz., last measured) condition is indicated when the arrow points to the left.

Even though not attaching (like KIRCHHEIMER, CASTLE, FREY—WYSS-

LING a.o.) great importance to protoplasmic streaming in the determination of the spiral growth of the cell wall as was done by VAN ITERSON and OORT—ROELOFSEN, it might be possible that the varying velocity of the stream in different parts of the sporangiophore plays a secundary part in the spiral growth of the cell wall. For this variable velocity, provided that this is also present under natural conditions, may give an explanation of the great individual differences in rotation found again and again by OORT—ROELOFSEN, CASTLE a.o., supposing that the protoplasmic stream, notwithstanding its parallel direction in relation to the long axis of the sporangiophore, plays after all a part in directing the micellae or the molecules.

Another question, according to GREHN, is the problem whether protoplasmic streaming is an autonomous phenomenon or a phenomenon that might be explained "durch eine osmotisch erklärbare Druckströmung und die Transpirationssaugung". There exist several arguments for the first-mentioned opinion. Primo; in a sporangiophore, which is surrounded by water under a cover glass, the evaporation cannot be the cause of a directed current and a fortiori not of a current in two opposite directions, as is the case in Phycomyces. In the second place the osmotic pressure from the mycelium in the substrate cannot be the only reason of the protoplasmic stream as this pressure can be released by cutting off the sporangiophore from the substrate without much influencing the velocity of streaming in the two opposite directions. For this purpose a series of velocity-measurements were performed both with sporangiophores which were still connected with the substrate and also with the same sporangiophore but now cut off from the substrate; even with cuttings from the middle of the sporangiophore, which were at times only 2 mm long and of which the protoplasmic stream persisted during several hours in the two directions. From these observations some may be cited here, which perhaps allow a conclusion e.g. that a streaming in the cell sap does not always influence the velocity of the protoplasmic layer, even not of the protoplasmic layer which borders directly to the cell sap. The velocity was measured in a sporangiophore, still connected with the mycelium, at 9.2 sec. (over 29 μ) towards the substrate and 11.1 sec. towards the spore-mass. Separated from the substrate the protoplasmic velocity was measured after five minutes at 9.4 sec. towards the substrate and 11.3 sec. towards the top notwithstanding the fact that protoplasmic contents of the sporangiophore were emptying through the cell sap. The passing of spherical masses of protoplasm throughout the cell sap was found frequently, as also stated by KIRCHHEIMER; and although these masses had apparently the same diameter as the vacuole no disturbance of the velocity could be seen in the protoplasmic layers. I should like to mention briefly a series of measurements, taken from a larger set of observations. From these measurements two facts become apparent; 1) that in the stream towards the substrate irregularities appear later than in the stream towards the spore-mass, viz.

the irregular acceleration of the microsomes over a distance of 2– ·5 μ, and 2) that a zigzag streaming of the microsomes, which runs towards the substrate, is caused by a honeycomb-pattern of the protoplasm, which appears first in the current towards the top, a phenomenon perhaps indicated by KIRCHHEIMER (p. 581).

TABLE IV.

Sporangiophore with a spore-mass, 3.8 cm long, 140 μ thick
All measurements were done 4 mm underneath the spore-mass

	Streaming velocity (time in sec./29 μ)		
	towards the substrate	towards the spore-mass	
2.05 p.m.			A piece of 9 mm was cut from the sporangiophore, measured from the top of the sporangium
2.10 p.m.	9.0	12.7	
2.25 „	8.9	13.1	
2.40 „	12.0	14.8	
2.55 „	13.5	16.0	
3.10 „	12.4	15.3	Many irregularities towards top, each leap 2—5 μ
3.25 „	11.9	15.4	Many irregularities towards top
3.40 „	12.5	14.2	Many irregularities towards top, Streaming towards top in a honeycomb-pattern
3.55 „	8.9	12.1	Many irregularities towards top. Towards the substrate also more zigzag streaming following the honeycomb-pattern of the stream towards the top
4.10 „	11.0	15.1	Towards top almost only irregularities
4.25 „	11.6	15.8	Towards top only one measurement
4.40 „	12.4	?	Towards top no measurement. Only „directed" Brownian movement towards top
4.55 „	13.5	14.0?	Towards top almost only Brownian movement
5.10 „	12.2	?	Towards top only Brownian movement

It is possible that the two facts, mentioned above, indicate a more important influence of the current towards the top, compared to the current towards the substrate. The question arises (in the supposition that the protoplasmic stream plays after all a part in directing the micellae) whether it is not possible that a change in the viscosity, as the sporangiophores grow older, causes a zigzag streaming of the protoplasm, which could partly explain the greater inclination of the micellae, which

according to OORT—ROELOFSEN occurs in the third layer of the cell wall. The asymmetry, moreover, of the acetyl-glucosamin molecules, by which the centre of gravity also has an asymmetric position in the molecule, could be perhaps another factor in the spiral growth of the cell wall, notwithstanding the fact that the current of the protoplasm runs usually parallel to the long axis of the cell. In the cited series of observations with a sporangiophore, which was removed from the substrate, it struck me that the current towards the top of the sporangiophore stopped earlier than that towards the substrate. In connection with the supposed autonomous character of the protoplasmic streaming the suggestion seemed obvious that a lack of material at the open end was the reason that the current towards the top stopped first, while the normal path of the microsomes into the outer stream at the top supplied more material in this outer stream, so that this streaming remained for a longer time. For testing this suggestion five series of observations were made, but now with pieces of sporangiophores which were cut from the middle of the sporangiophore (lengt 6 cm, diameter 60—70 μ) at a distance of 3—1½ cm from the top of the sporangiophore. The pieces had a length respectively of 1.3; 0.5; 0.25; 0.225 cm. Five minutes after the cutting the velocities were measured $\pm$ in the middle of the piece with interval of ten minutes, until the streaming stopped in both directions .From these data were calculated the mean times, mentioned below, necessary to cover a distance of 29 μ. The distance covered was calculated from the time which passed until the current stopped in the two directions, supposing that the velocity towards the top of the sporangiophore diminishes as much as this increases towards the substrate, a supposition which is certainly not true at the end of the series of observations, when the velocity at the top is much smaller than towards the substrate.

TABLE V.

Length of the piece	Average time in sec/29 μ		Time of streaming in minutes	Calculated distance in cm	
	Towards the substrate	Towards the spore-mass		Towards the substrate	Towards the spore-mass
1.3 cm	8.8	12.6	140	2.8	1.9
0.5 „	12.1	16.1	60	0.88	0.62
0.25 „	13.8	17.1	50	0.63	0.52
0.25 „	12.6	16.1	45	0.62	0.48
0.225 „	13.8	17.1	65	0.82	0.66

Though the distances covered by both currents, calculated in this way, are much longer than the length of the pieces used, I do not believe that a transition of microsomes occurred from the inner layer of the protoplasm

into the outer one, for such a transition of microsomes was never observed at the end of the experiment, except in the last series with the piece of 0.225 cm, wherein perhaps such a transport was seen on the side where the spore-mass had been. This, however, could also be explained by the often-observed direct transition in undamaged sporangiophores. In my opinion it is most evident that one must look for the cause of the protoplasmic streaming in the protoplasm itself, and not in a boundary-surface phenomenon e.g. a) between protoplasm and outer wall, for then it is difficult to understand why the inner stream persists in pieces of a sporangiophore or b) between protoplasm and cell sap, for how would it be possible then that 1) big masses of protoplasm, which pass through the cell sap, have no influence on the inner streaming and 2) that the outer streaming persists in cut sections of the sporangiophores.

Possible explanation.

In relation to the supposed connection between protoplasmic streaming and spiral growth of the cell wall, some suggestions may be proposed here, keeping, however, in mind the purely hypothetical character of these suggestions.

Because no spirally-streaming protoplasm was ever observed in the material used, first of all the spiral growth of the material was tested. For this purpose several sporangiophores, both of the "+" and "—" strain, were fitted in diffuse light with moist grains of "Norit" on one side of the spore-mass and after this treatment placed in the dark. Though this experiment was too superficial to measure definite angles of rotation, in both strains a rotation of the spore-mass could be observed. A nutation as suggested by CASTLE seemed to me less probable, where an angle of rotation of 160° (or 160° + 360°) was found after 2½ hours. So spiral growth of the cell wall occurred in the material used, notwithstanding the fact that never spirally streaming protoplasm was observed. I should like to suggest a hypothesis in relation to the work of VAN ITERSON about the structure of the cell wall of Valonia; for the elastic forces — to which CASTLE ascribes a possible rôle in effecting the spiral structure of the primary wall of Phycomyces, — cannot be the only reason of spiral growth (whereat CASTLE hints himself in a footnote), when a left-hand spiral (defined as spiralling in the direction of the thread on a left-hand screw) occurs more times than a right-hand spiral. The hypothesis would be that three factors cooperate; a) the plastic- and elastic properties of the cell wall, b) the asymmetry of the acetyl-glucosamin molecule and c) the changing velocity of protoplasmic streaming caused by the difference in thickness of the protoplasmic layer at different height in the sporangiophore and by the age of the cell.

One could imagine [1]) that the acetyl-glucosamin molecule is flat and moreover longer than broad, while the acetyl-unit, placed at one side, would make the molecule biased on that side. In this way one would obtain a modification of the suggestion made by FREY—WYSSLING, p. 131 [2]), in which, however, the fluctuation of the angles of spiralling becomes more plausible. This hypothesis might give at the same time an explanation, opposed to the one given by VAN ITERSON, of the periodical change in direction of the crystallites in two subsequent lamellae, by the assumption that the asymmetric units of one and the same layer form together submicroscopic ridges, along which the molecules of the subsequent layer glide in such a way that the asymmetric units of the subsequent layer repeat this process in the opposite direction.

By means of our hypothesis which connects protoplasmic streaming with asymmetric structure of the molecule, one could explain the difference in structure of the primary, secondary and tertiary wall of Phycomyces (in the sense of OORT—ROELOFSEN) as follows:

1. In the growth-zone [3]) the molecules, in forming the primary wall, should move forward on their flat side, but much deviated from the direction parallel to the long axis by the asymmetry of the acetyl-unit. The orientation of the chitin molecule will not be, however, perpendicular to the direction of the streaming because the acetyl-unit is not fastened to the chitin in its centre.

2. At a greater distance from the spore-mass, where the secondary wall is formed, the protoplasmic streaming itself is more parallel to the long axis of the cell, while the much greater streaming velocity produces a far better orientation of the molecules with their long axes almost parallel to the direction of the stream. A perfect parallelism should not occur on account of the asymmetry of the molecule. The difference in velocity of the protoplasmic streaming caused by individual differences of the sporangiophores (by difference in age of the same sporangiophore or by the difference in thickness of the protoplasmic layer according to a different distance from the top in the same sporangiophore) could in

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[1]) Provided that this possibility is not already excluded by the results of MEYER–PANKOW and ITERSON–MEYER–LOTMAR, as e.g. MEYER–PANKOW write on p. 594; "Nous sommes donc en droit de considérer le groupe constitué par les deux restes d'acétyle-glucosamine, que nous appellerons chitobiose, comme unité de structure", because in that case the asymmetry of the acetyl-glucosamin molecule is already balanced out in his model of chitin (of Palinurus vulg. just as in the second publication of Phycomyces). My ignorance of röntgenographic methods does not allow me to form a clear opinion in this matter.

[1]) "Ein weiteres Problem bilden die Beziehungen zwischen Protoplasmaströmung und Schraubenstruktur. Es gibt in der Natur viel organische Stoffe, die von sich aus zu schraubigen Kristallaggregaten heranwachsen".

[3]) In the growth-zone the streaming velocity of the protoplasm is very small and moreover the direction parallel to the long axis of the cell is much disturbed by the greater number of microsomes and by the Brownian movement.

our supposition give a ready explanation of the difference in the angles of rotation of different sporangiophores or of the same sporangiophore at subsequent times.

3. In the formation, finally, of the tertiary wall the velocity of the streaming, much diminished by age, would result again in a decrease of the directive force of the streaming upon the molecules. Ageing also may cause, as observed above, a "honeycomb-pattern" in the protoplasm, which would be in agreement with the description of OORT—ROELOFSEN, of this tertiary layer; "Die Schicht zeigt eine ziemlich grobe netzförmige oder schachbrettartige Zeichnung."

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Palaeontology. — *On cretaceous Nerinea's from Cuba.* By H. KNIPSCHEER. (Communicated by Prof. L. RUTTEN.)

(Communicated at the meeting of May 28, 1938.)

In the collections, made by the Utrecht Cuba Expedition 1933 there were still some well-preserved *Nerinea*'s from Southern Santa Clara and Camaguey, which hitherto not had been studied. Equally undetermined were some *Nerinea*'s from Camaguey, collected in 1933 by Dr. TSCHOPP (of the Bataafsche Petroleum Maatschappij) in Camaguey, and presented by him to the Utrecht Geological Institute. In the following a short description will be given of four species.

Nerinea bicincta Bronn. fig. 1, 2, 3*a*, *b*, *c*.

H. G. BRONN, Neues Jahrb. 1836, p. 562, pl. VI, fig. 14; GOLDFUSS, Petref. Germ. 1844, 3, p. 46, pl. CLXXVII, fig. 5*a*, *b*.

Conical, with rather low convolutions; one row of knots with twelve knots on each winding; the knots of the different convolutions connected with each other in vertical sense. Four infoldings: two from the columella, one from the inner lip and one from the outer lip. Our specimens agree externally and internally with the species described by BRONN.

Localities: *a*. 700 m. S. from Aurora, Camaguey; *b*. Cantera Caballero, W. from Sibanicu, Camaguey.

Habana formation (Maestrichtian). The species, described by BRONN is from the Gosau-Cretaceous and from the "Upper Quader-Sandstein" (Senonian).

Nerinea (Plesioptygmatis) burckhardti Böse. fig. 4, 5.

E. BÖSE, Bol. Inst. Geol. Mexico, 24, 1906, p. 66—67, pl. XV, fig. 3—13.

Our specimens only rarely show any detail of the ornamentation, there only being visible an indistinct suture at the distal end of some of the convolutions. I did not detect a second spiral line, which BÖSE mentions from the proximal part of each winding. The deepest part of each convolution lies near the distal end. The height of the convolutions is rather variable: one specimen having 5 windings on 3.5 cm., another one of almost the same width 5 windings on 2.9 cm. There are no complete specimens. Four infoldings: two from the columella, of which the distal one is the deepest, one from the innerlip and a small one from the outer lip.

Localities: *a*. Ingenio Grande, Camaguey, SW. from Camaguey City; *b*. San José de los Jibaros, Camaguey.

Habana formation (Maestrichtian). BÖSE's specimens are from the

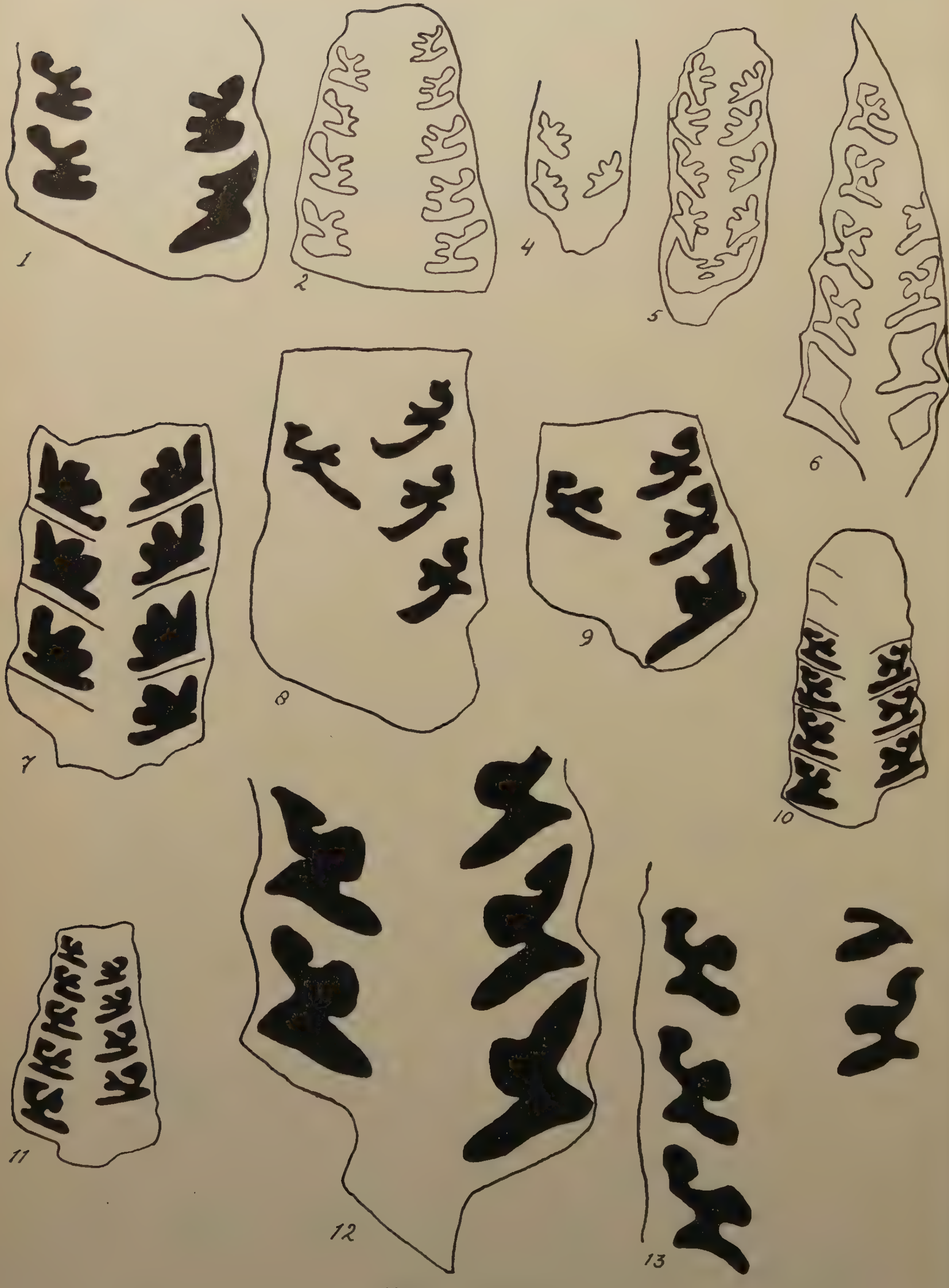

Fig. 1, 2 and 4—13.

Cardenas beds; from a niveau, lower than marls with Orbitoides. According to MUIR and MAC GILLAVRY [1]) the Cardenas beds are Maestrichtian.

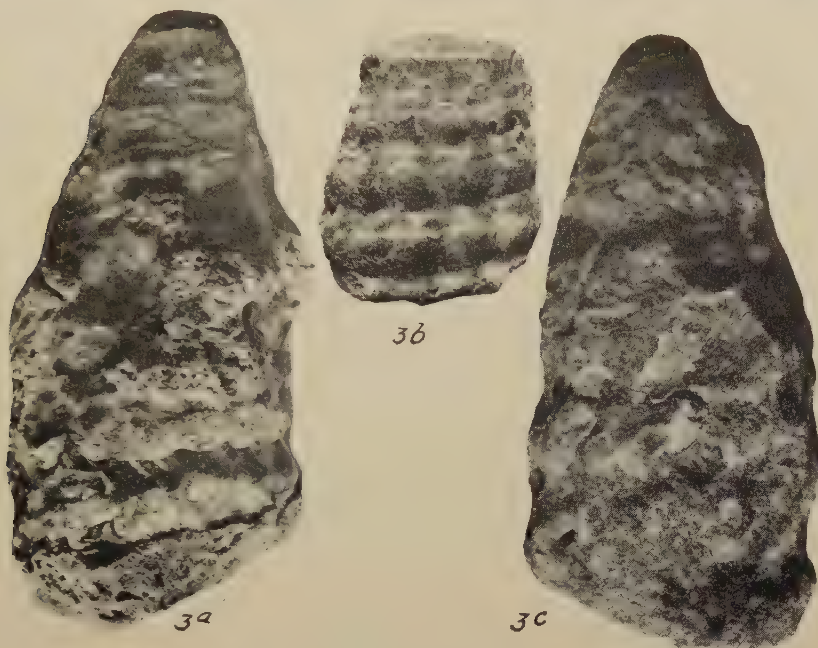

Fig. 3a—3c.

Ptygmatis forojulensis Pir. fig. 6—11.

M. PIRONA. Mem. R. Inst. Veneto Sc. etc. 22, 1883, p. 162, pl. IIa, fig. 1—5; C. F. PARONA. Mem. Carta. Geol. Ital. 5, 1, 1909, p. 214, pl. XXV, fig. 7—20; K. FUTTERER, Pal. Abh. 6, 1, 1892, p. 112, pl. XI, fig. 8, 9; G. BÖHM, Palaeontogr. 41, 1895, p. 134, pl. XIII, fig. 5, 6.

Different Nerinea's, rather divergent inter-se have been described under this name. Especially the internal structures are very different. PIRONA (our fig. 7) describes four infoldings: two from the columella, a deep one from the inner lip and a small one from the outer lip. In PARONA's figure (our fig. 11) the infoldings are more pronounced, especially the proximal columella fold and the one from the outer lip. In some of PARONA's figures (our fig. 10) there are even more than four folds, whereby the fossils approach the genus Ptygmatis. In FUTTERER's and BÖHM's specimens the four infoldings are deeper and moreover oblique (our fig. 8 and 9), while at the same time a feeble fifth infolding at the roof occurs. Our fossil agrees best with the specimens of PARONA (our fig. 6 and 11).

Locality: South from Provincial, Southern Santa Clara, Cuba.

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1) J. M. MUIR, Geol. of the Tampico Region, Mexico (1936).
H. J. MAC GILLAVRY, Geology of the province of Camaguey, Cuba etc. Utrecht, Acad. Thesis (1937).

Provincial Limestone of Cenomanian-Turonian or Albian age [1]). According to PARONA the age of *Nerinea forojulensis* is Cenomanian.

Nerinea cf. *gigantea* d'Hombr. Firmas. fig. 12, 13.

A. D'ORBIGNY, Pal. française. Terr. Crét. 2, 1842, p. 77—78, pl. CLVIII, fig. 2.

As to form and dimensions our specimens agree well with *N. gigantea*. The folds of our specimens agree also well with the infoldings at the mouth of D'ORBIGNY's type specimen, but much less with sections of the older convolutions of this type specimen. Our specimens have three infoldings: one from the columella, one from the inner lip and one from the outer lip. There is moreover a trace of an infolding at the proximal side of the convolutions. Transversal sections through the last winding in one individual 5½ cm., in the smallest one still 4½ cm.

Locality: San Cristobal, S. of Seibabo, Southern Santa Clara, Cuba.

Provincial limestone (Cenomanian-Albien[1]). The age of N. gigantea in Europe is Urgonian (Barrêmian-Aptian).

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[1]) A. THIADENS, Geology of Southern Santa Clara Province, Cuba. Utrecht, Acad. Thesis (1937).

H. J. MAC GILLAVRY l.c.

Medicine. — *Das Exochorion der Stegomyia-Eier.* Von A. DE BUCK. (Zoological Laboratory, Department of Tropical Hygiene, Royal Colonial Institute, Amsterdam). (Communicated by Prof. W. A. P. SCHÜFFNER.

(Communicated at the meeting of May 28, 1938.)

Die für die Vergleichung der *Anopheles maculipennis*-Rassen wichtige Zeichnung der Eier rührt bekanntlich vom Exochorion her, sie läszt sich nur vom feineren Bau des Exochorions aus verstehen (DE BUCK und SWELLENGREBEL, 1932). Schon damals ,als ich mit dem Studium dieses Exochorions von *A. maculipennis* beschäftigt war, konnte ich gelegentlich feststellen, wie wesentlich anders die Verhältnisse beim *Stegomyia*-Ei liegen. Es war mir aber nicht möglich genügende Zeit darauf zu verwenden um die Sache zur Klarheit zu bringen. Erst kürzlich habe ich wieder Gelegenheit gefunden die Eier von *Stegomyia fasciata* und *albopicta* (*Aedes aegypti* und *albopictus*) genauer zu untersuchen; die Resultate will ich hier mitteilen.

Dasz es nicht so ganz leicht ist, den Bau des Exochorions dieser Eier zu verstehen, mag daraus hervorgehen, dasz die Beschreibungen in der Literatur meist sehr oberflächlich, zum Teil sogar völlig unrichtig sind.

So sagen OTTO und NEUMANN (1905, S. 376): Die Eier zeigen punktförmige Sprenkelung, die sich bei starker Vergröszerung in bläschenartige Gebilde auflöst. Die Bläschen enthalten Luft....

JAMES und LISTON (1911, S. 5): The eggs of the genus *Stegomyia* are peculiar in that, besides being more or less oval in shape, they possess a rim of cells somewhat resembling the rim or frill present in anopheline eggs.

NEUMANN und MAYER (1914, S. 207): Bei stärkerer Vergröszerung beobachtet man eine Unzahl kleiner bläschenartiger Gebilde auf der Oberfläche, welche Luft enthalten und das Schwimmen auf der Wasseroberfläche ermöglichen.

HOWARD, DYAR und KNAB (1917, S. 836): Egg. — Fusiform, black, very slightly flattened on one side, slightly more tapered towards the micropylar end; sculpture of rough, somewhat irregular rhomboidal callosities forming spiral rows.

PATTON und EVANS (1929, S. 255): The egg of *Stegomyia fasciata* is spindle-shaped, slightly asymmetrical and somewhat curved in outline, it is of a light grey colour when first laid, but soon turns black. The surface of the chorion has characteristic sculptured markings, consisting of reticulated, polygonal areas, the spaces between which are raised. The raised areas appear to consist of some substance, which has the effect

of preserving the eggs when exposed to unfavourable conditions, such as heat and desiccation.

ROUBAUD (1929, S. 1176): l'exochorion enveloppe l'oeuf uniformément d'un réseau polyédrique, d'apparence cellulaire, particulier. A un fort grossissement, ce réseau se montre constitué par des éléments cellulaires polygonaux pourvus au centre d'un épaississement en bouton, saillant à la surface de l'oeuf. Il paraît vraisemblable que cette structure est due à une véritable exfoliation de la muqueuse des voies génitales.

Mit den von OTTO, NEUMANN und MAYER genannten Luftbläschen sind wohl die Gebilde gemeint, die man bei durchfallendem Licht perlenschnurartig um den Rand des Eies herumgezogen sieht. Bei auffallendem Licht verschwinden diese und man sieht nur das schwarze Ei mit einer Anzahl von weiszen silberglänzenden Punkten besät, wenn nämlich das Ei auf dem Wasser schwimmt und die Oberseite nicht benetzt ist. Das *Stegomyia*-Ei unterscheidet sich nämlich vom *Anopheles*-Ei durch die Leichtigkeit, womit es zum Sinken gebracht wird. Wenn man ein *Anopheles*-Ei untertaucht, nimmt es eine grosze Luftblase mit, die an der Oberseite haften bleibt und es sofort wieder an die Wasseroberfläche bringt, wenn man das Ei sich selbst überläszt; das *Stegomyia*-Ei jedoch braucht man nur unter Wasser zu drücken und es wird von selbst weiter zum Boden sinken. Auch kann man gelegentlich in einem Gelege viele Eier haben, die nur noch mit den beiden Spitzen der ein wenig konkaven Oberseite an der Wasseroberfläche liegen, während die ganze übrige Oberseite sich unter Wasser befindet. Meistens ist die Oberseite der Eier jedoch nicht so ausgesprochen konkav dasz dies möglich wäre. Auch die an den Wänden des Gefäszes abgelegten Eier sinken meistens sofort, wenn man sie auf das Wasser bringen will. Darum können die Eier auch leicht einen asymmetrischen Eindruck machen (PATTON und EVANS). Die Eier, die man unter dem Mikroskop hat, liegen meistens mehr oder weniger auf der Seite, ohne das man sich dessen bewuszt ist. Auch die Beschreibung von HOWARD, DYAR und KNAB scheint mir hierauf keine Rücksicht zu nehmen. Es wird nur gesprochen von „slightly flattened on one side", nicht von Ober- und Unterseite. Betrachtet man die Oberseite der untergetauchten Eier, sieht man auf dem schwarzen Grunde anstatt der ebengenannten Punkte ein Netz von silberglänzenden Linien. Die Maschen des Netzes sind mehr oder weniger sechseckig. Auch die Unterseite der schwimmenden, wie der untergetauchten Eier zeigt ein derartiges Silbernetz, wenn auch die Maschen eine viel mehr längliche Form haben, rechtwinklig zur Längsachse des Eies. Auch hier läszt die Vergleichung mit den *Anopheles*-Eiern im Stich. Bei diesen ist die ganze Unterseite silberglänzend, weil sie von einer Luftschicht überzogen ist, die von den zahllosen winzigen Höckern des Exochorions festgehalten wird. Zwar zeigt auch hier die Unterseite eine netzartige Zeichnung, aber diese wird durch die netzartige Anordnung von Höckern verursacht, die ein wenig gröszer sind als die andern. Immerhin kann man vermuten, dasz

der Silberglanz auch bei den *Stegomyia*-Eiern von der Anwesenheit von Luft herrührt. Diese Vermutung wird noch verstärkt, wenn man sieht wie die Silberlinien sofort in Alkohol verschwinden, gerade wie auch *Anopheles*-Eier in Alkohol ihre Zeichnung verlieren und schwarz werden, weil das Exochorion in sich völlig durchsichtig ist und das schwarze Chorion nicht verdecken kann.

Dasz wir es hier tatsächlich mit einem Netz von Luftadern zu tun haben, sieht man deutlich an dem abgelösten Exochorion oder an einem frisch abgelegten, noch nicht geschwärzten Ei. Es gelingt nämlich ziemlich leicht ein Stück des Exochorions vom Ei abzuheben, wenn man dasselbe auf dem Objektträger antrocknen läszt. Entfernt man dann mit einer Nadel das Ei, so bleibt oft ein Fetzen des Exochorions an dem Glas haften. In Fig. 9 sieht man so ein Stück des Exochorions der Oberseite, in Wasser unter dem Deckglas photografiert. Noch vollkommener gelingt die Isolierung des Exochorions, wenn man das Ei, oder besser eine leere Eischale, zwischen Objektträger und Deckglas trocknen läszt. Da bekommt man oft die ganze Oberseite auf dem Deckglas und die Unterseite auf dem Objektträger. In diesem Fall ist aber meistens keine Luft mehr vorhanden (Fig. 10, 11).

Auch läszt sich das Exochorion leicht vom Ei abheben, wenn man es einige Zeit in verdünntem Natriumhypochlorit hat liegen lassen. Dies hat eine auflösende Wirkung auf das Exochorion, wie bekanntlich ROUBAUD gezeigt hat, der es in der Verdünnung von 1 : 1000 brauchte um die *Stegomyia*-Eier zum Schlüpfen zu bringen. Diese aktivierende Wirkung wird nach ROUBAUD dem Natriumhypochlorit gerade von der auflösenden Kraft verliehen. Läszt man es nicht zu lange einwirken und legt man dann ein Deckglas auf, kann man das halb aufgelöste Exochorion sich vom Ei abheben sehen, als eine glashelle, homogene Haut, nur von dem Luftgeäder durchsetzt; alles andere ist aufgelöst. Weil ich anfangs nur in dieser Weise das Exochorion zu Gesicht bekam, hatte ich stark den Eindruck, dasz diese Luftadern tatsächlich Luftröhrchen, eine Art von Tracheen seien. Es hat mir viele Mühe gekostet um mich davon zu überzeugen, dasz diese Vorstellung falsch ist und dasz wir es hier nur mit in offenen Rinnen festgehaltener Luft zu tun haben. Vornehmlich die Untersuchung von frisch abgelegten, noch nicht geschwärzten Eiern hat hier gute Dienste geleistet.

Bei der Deutung der verschiedenen Bilder war noch eine andere Schwierigkeit. Nicht alle Eier zeigen die anfangs erwähnte Perlenschnur. Viele Eier, in einigen Gelegen alle, besitzen statt deren an einer oder beiden Seiten einen glashellen, homogenen Saum, oft besonders deutlich zu sehen als eine Art von Kiel bei den Eiern, die auf der Kante liegen (Fig. 1). Untersucht man die Sache näher, am besten bei mäszig plattgedrückten leeren Eischalen, so sieht man folgendes:

Die Oberseite zeigt immer dasselbe Bild (Fig. 2). Die Unterseite, wenn eine Perlenschnur anwesend ist, zeigt Fig. 3. Das Exochorion trägt

Fortsätze, Säulchen, wie bei *A. maculipennis*, die aber nicht frei stehen, sondern von einer Art Blasen umgeben sind[1]). Die Luftadern laufen

Fig. 1. Auf der Seite liegendes Ei mit „Saum".

Fig. 2. Oberseite des Eies, Knöpfe von der Seite gesehen.

Fig. 3. Unterseite des Eies, Blasen und Säulchen von der Seite gesehen.

zwischen den Blasen, man sieht sie am Rande in den Tälern umbiegen (Fig. 4). Die Unterseite, wenn ein Saum da ist, zeigt Fig. 5. Die Säulchen

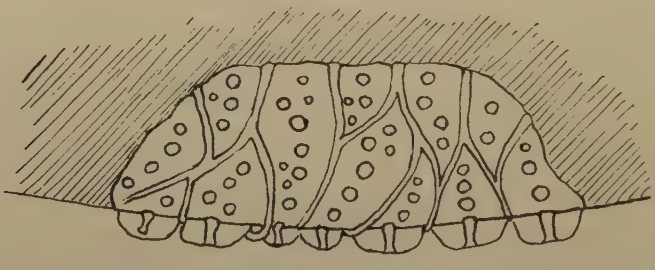

Fig. 4. Exochorion der Unterseite; ein Teil desselben, der hier gezeichnete Teil, nach oben umgeklappt; das untere Blatt nur schraffiert angedeutet.

Fig. 5. Unterseite des Eies, Saum und Säulchen von der Seite gesehen.

[1]) Aus diesem Grunde und auch weil sie auf der Unterseite und nicht, wie bei Anopheles, auf der Oberseite vorkommen, kann man diese Fortsätze wohl nicht für ganz homolog mit den Columellae der Anopheles-Eier halten. Darum habe ich sie hier Säulchen genannt.

stehen frei auf dem Exochorion, die Luftadern sieht man unter dem Saum hindurch umbiegen (Fig. 6).

Auch hier hat die Untersuchung von frisch abgelegten Eiern und aus

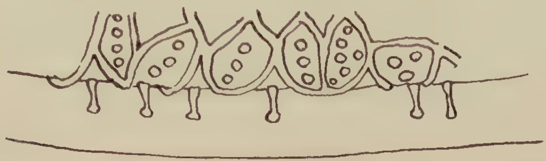

Fig. 6. Exochorion der Unterseite mit Saum, an der Umbiegungsstelle gesehen.

dem Follikel freipräparierten reifen Eiern die Lösung der Frage gebracht. Ich will nun in aller Kürze den Bau des Exochorions beschreiben.

Auf der Oberseite des Eies trägt das Exochorion kurze, gedrungene Fortsätze, runde Knöpfe, die ziemlich lose auf dem Exochorion stehen.

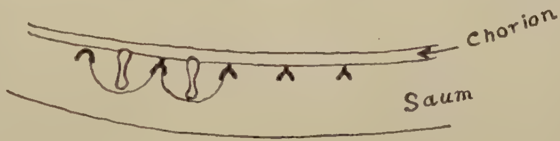

Fig. 7. Reifes Ei aus dem Ovar in destilliertem Wasser unter Deckglas plattgedrückt. Unterseite des Eies mit noch nicht ganz verschleimtem Exochorion. Weil der Saum optischer Durchschitt des Schleimmantels ist, kann er sich hier über den zwei ungeschädigten Blasen fortsetzen.

Sie stellen die hellen, silberglänzenden Punkte dar, wenn man das auf dem Wasser schwimmende Ei mit auffallendem Licht betrachtet. Die Oberseite dieser Knöpfe ist nicht eben, sondern von untiefen Furchen in

Fig. 8. Frisch abgelegtes Ei in destilliertem Wasser, die Rinnen schon mit Luft gefüllt. Unterseite des Eies, das Exochorion an der rechten Seite verschleimt.

Höcker zerlegt. Sie sind von einem Kranz von kleinern Fortsätzen umgeben. Zwischen diesen letztern läuft die präformierte Rinne, worin die Luft nach dem Untertauchen des Eies fest gehalten wird (Fig. 9, 10).

Auf der Unterseite des Eies trägt das Exochorion schlanke Fortsätze, Säulchen, die aber nicht frei auf dem Exochorion stehen, sondern in Blasen eingeschlossen sind. Die Form dieser Blasen ist auf der Abbildung gut zu sehen (Fig. 11). In jeder Blase ist meist eine Reihe von Säulchen. Schnitt-

präparate von dem Ovar zeigen dasz die Kerne des Follikelepithels in den Tälern zwischen den Blasen liegen, wie auf der Oberseite zwischen den Knöpfen. Die zelluläre Struktur des Exochorions ist also nicht einfach ein Abdruck der Follikelepithelzellen, wenn sie auch von diesen bedingt ist. In den Tälern laufen auch die Luftrinnen. Untersucht man ein reifes Ei aus dem Ovar in physiologischer Kochsalzlösung, so bleiben die Blasen ungeschädigt, auch wenn man das Deckglas fest andrückt. Ersetzt man aber die Salzlösung durch Brunnenwasser, sieht man nach einigem Warten die Blasen aufquellen, die Auszenwand wird unscharf und im nächsten Moment sind alle Blasen zu einem einheitlichen Schleimmantel verquollen, der sich in optischem Durchschnitt wie ein Saum ausnimmt. Sehr instruktiv ist die Fig. 7, wo ich den Augenblick getroffen habe, wo ein Teil der Blasen schon verquollen war, ein andrer Teil noch nicht. Hier sieht man deutlich in den Tälern eine präformierte Stelle der Auszenwand, wo die aufquellende Wand abbrechen wird. Nach dem auflösen der Blasen bleiben diese präformierten Stellen übrig, es sind eben die Rinnen in optischem Durchschnitt. Ueber sie hinweg zieht sich der Schleimmantel. Deutlich ist dies auch beim frisch abgelegten Ei, wo die Rinnen schon mit Luft gefüllt sind. Auch hier kann man die Blasen zum Verquellen bringen, man sieht dann, wie sie über den Luftadern in einander flieszen (Fig. 8). Es scheint, dasz der Druck dazu kommen musz um die Verschleimung der Blasenwände zu bewirken, ohne Deckglas bleiben die aus dem Ovar freipräparierten Eier auch in destilliertem Wasser meist ungeschädigt. Doch musz auch oft diese Verschleimung bei vielen Gelegen spontan auftreten; das sind dann die Eier, wo wir einen Saum oder Kiel zu sehen meinen (Fig. 1). Ist das Ei einmal ausgeschwärzt, so hat der Druck keine Verschleimung mehr zur Folge.

Betrachtet man das abgelöste Exochorion der Unterseite, sieht man oft nur die runden Tüpfel der Säulchenkapitelle (Fig. 11). Meistens aber haben die Säulchen eine schlankere Form und sieht man sie bei genügendem Druck auf dem Deckglas in liegender Haltung. Uebrigens ist die Form der Säulchen in den verschiedenen Partieen des Exochorions eine verschiedene. In den an den Knöpfen der Oberseite grenzenden Partieen ist die Form eine mehr gedrungene, um allmählich in die schlanke Form der mittleren Unterseite über zu gehen (Fig. 11, oben). Der einzige Unterschied zwischen den Eiern von *albopicta* und *fasciata* scheint in der geringeren Grösze der *albopicta*-Eier zu liegen; im Bau des Exochorions sind die zwei Arten einander völlig gleich.

Wie aus alle dem hervorgeht besteht ein groszer Unterschied zwischen dem Exochorion der *Stegomyia*-Eier und dem der *Anopheles*-Eier, namentlich was die Unterseite des Eies betrifft. Man wird wohl nicht fehlgehen, wenn man dies in Verbindung bringt mit dem verschiedenen Verhalten dieser Eier der Austrocknung gegenüber. Während die *Anopheles*-Eier rasch nach dem Trocknen zusammenschrumpfen, können die *Stegomyia*-Eier eine Austrocknung ohne jede Schrumpfung ertragen.

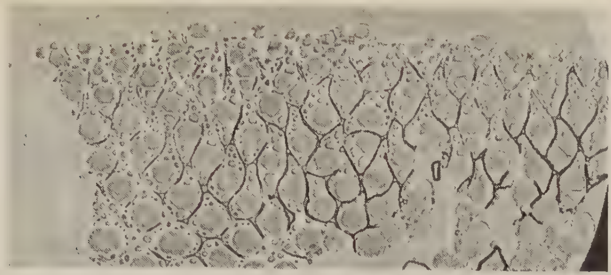

Fig. 9. Exochorion der Oberseite. Rinnen mit Luft gefüllt. Vergr. 285 ×.

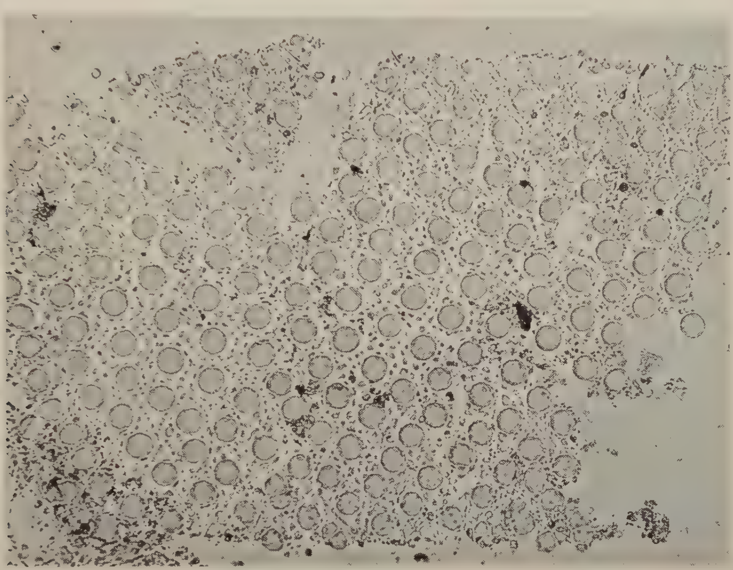

Fig. 10. Exochorion der Oberseite. Rinnen ohne Luft. Vergr. 285 ×.

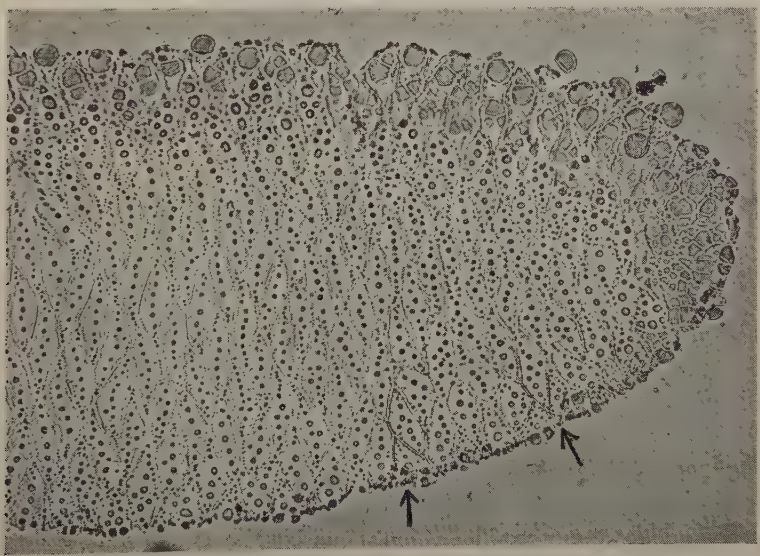

Fig. 11. Exochorion der Unterseite. Bei den Pfeilen noch einige Rinnen mit Luft gefüllt. Vergr. 285 ×.

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Medicine. — *Direkte Endophotographie.* Von J. LUBBERS. (Communicated by Prof. A. DE KLEYN.)

(Communicated at the meeting of May 28, 1938.)

Unter direkter Endoskopie versteht man ein Verfahren, bei welchem die inneren Organe ohne Benutzung des Spiegelbildes besichtigt werden können. Das Photographieren dieser Organe durch die bei der direkten Endoskopie gebrauchten Röhren möchte ich dementsprechend „direkte Endophotographie" nennen.

Die verschiedensten Beleuchtungssysteme sind bisher bei der direkten Endoskopie zur Anwendung gekommen. Die Lichtquelle kann distal oder proximal angebracht werden. In Amerika wird die distale Methode benutzt (EINHORN, JACKSON, INGALS, MOSHER, u.a.). Als Lichtquelle gebrauchte man zuerst einen von MIKULICZ in der Röhre angebrachten Platinglühdraht, welcher jedoch später durch eine von EDISON erfundene Mikrolampe ersetzt wurde.

In Europa gebraucht man meistens die proximale Beleuchtung verschiedener Systeme (CASPER, GOTTSTEIN, BRÜNINGS, KAHLER–LEITER, HASLINGER, WALDAPFEL, BOULET, u.a.).

In England wendet V. E. NEGUS in seinem System eine Kombination der distalen und proximalen Beleuchtung für das Broncho-elektroskop an, während er für das Ösophagoelektroskop eine doppelte Lichtquelle benutzt. Für diese letztere Beleuchtung ist eine ziemliche Erweiterung eines grossen Teiles der Untersuchungsröhre notwendig; daher ist sie für die engeren Bronchoskopieröhren unbrauchbar.

Für genauere Orientierung über diese Systeme möge auf die verschiedenen Beschreibungen in der Literatur hingewiesen werden. Nur sei hier noch bemerkt, dass bei den meisten Beleuchtungssystemen die Optik den Eingang der Röhren immer mehr oder weniger verkleinert.

Bei dem HASLINGER'schen Elektroskop befindet sich die Optik ganz oberhalb des Niveaus der Röhre (Schema 1).

Neu waren hierbei 1. die Erweiterung des proximalen Teiles der Untersuchungsröhren (*c*—*d*), um das von seitwärts einfallende Licht nicht zu verringern und 2. eine Kreuzung in dem reflektierten Licht. HASLINGER sagt, dass er sein Beleuchtungssystem (Lampe, Linse und Spiegel) derartig gewählt hat, dass der Lichtbüschel an der Stelle, wo er in die Röhre eintritt, beinahe den gleichen Durchschnitt hat wie die Erweiterung *c*—*d*. Der Lichtbüschel bleibt hier also gleich gross. HASLINGERs Beleuchtungsprinzip bedeutet einen wichtigen Schritt vorwärts auf dem Gebiete der

direkten Endoskopie. Ein Teil des Lichtes wird nicht gegen die Röhrenwand reflektiert (*e*).

Schema 1.

r. Untersuchungsröhre.
h. Glühdraht.
a. Linse.
b. Hohlspiegel.
c—*d*. Proximale Erweiterung der Röhre.
g. Hauptachse des Lichtbüschels.
e. Direktes Licht.
f. Reflektiertes Licht.

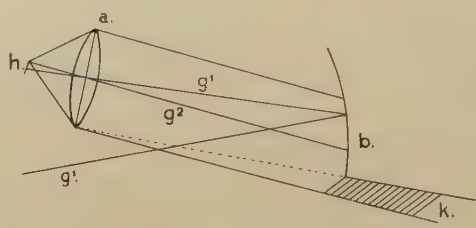
Man könnte sich die Beleuchtung als eine Lichtquelle im Eingange der Röhre vorstellen. Diese Lichtquelle wird dem Auge keine Beschwerden verursachen. Sie wird dann von dem Bilde des Glühdrahts, das die Optik im proximalen Teil der Röhre hervorruft, gebildet.

Dieses Instrument hat aber doch einige Nachteile. Erstens muss der Spiegel *b* (Schema 1) verstellbar sein. Mittels zweier Schrauben kann man demselben eine derartige Stellung geben, dass man stets eine optimale Beleuchtung am Ende der Röhre bekommt. Notwendig ist, bei jeder Untersuchung vorher eine solche Stellung zu ermitteln. Dies muss zuweilen während der Endoskopie wiederholt werden. Ich habe nämlich bemerkt, dass sich die Stellung des Spiegels während der Untersuchung ändern kann. Auch beim Gebrauch von Verlängerungsröhren kann es nötig sein, den Spiegel wieder einzustellen, da man dann eine grössere Entfernung zwischen dem distalen Röhrenende und der Optik erhält.

Zweitens ist es ein Übelstand, dass das Licht unter *b* durch, auf das Gesicht des Untersuchers fällt. Dieses Licht ist als ein starker Lichtfleck auf dem medialen Teil des unteren Augenlides zu sehen. Dies wird wenn auch nicht von allen, so doch von vielen Untersuchern als störend empfunden. Ausserdem kann eine kleine Bewegung des Untersuchers bewirken, dass das Gegenlicht auf den Augapfel fällt. Weiter habe ich bemerkt, dass das Gegenlicht beim Photographieren durch die endoskopische Röhre sehr störend ist.

Die Entstehung dieses Gegenlichtes ist, obgleich HASLINGER niemals eine vollständige Konstruktion des ganzen Lichtbüschels gegeben hat, mit ziemlich grosser Sicherheit festzustellen. Der Glühdraht seiner Lampe befindet sich wahrscheinlich in der Brennfläche der Linse *a*, sodass ein Lichtbüschel, welcher anscheinend möglichst parallel ist, auf den Spiegel *b* fällt. Dieser Lichtbüschel ist aber in Wirklichkeit nicht parallel, sondern etwas streuend, da wir nicht mit einem Lichtpunkt, sondern mit einem

Glühdraht zu tun haben. Hierdurch wird die seitliche Begrenzung des Lichtbüschels von Licht gebildet, das mit den Nebenachsen der äussersten Lichtpunkte des Glühdrahtes parallel läuft (Schema 2).



Schema 2.

h. Glühdraht.
a. Linse.
b. Spiegel.
g^1. Hauptachse des Lichtbüschels.
g^2. Nebenachse des Lichtbüschels.
k. Gegenlicht.

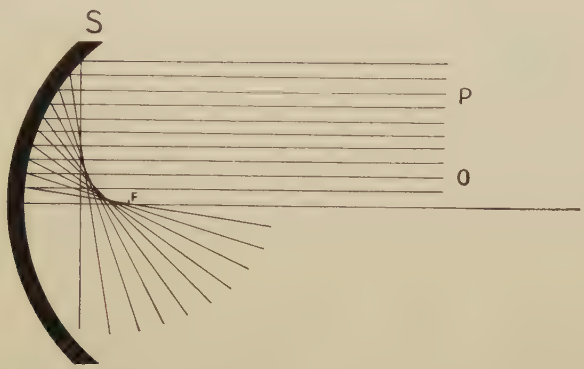
Die äusserste Begrenzung fällt nun auf das Gesicht des Untersuchers. Durch Verlängerung des Spiegels *b* nach unten wäre es möglich, dies zu vermeiden. Das Fehlen technischer Hilfsmittel verhinderte mich, eine derartige Konstruktion zu versuchen. Da HASLINGER aber seinen Spiegel mit Recht oberhalb des Niveaus der Röhre behalten will, muss man den ganzen Beleuchtungsapparat nach oben verlegen, um dies zu erreichen. Man kann dann erwarten, dass die Beleuchtung geringer wird, weil die Hauptachse des reflektierten Lichtbüschels jetzt einen grösseren Winkel mit der Röhre bilden muss, damit dieser Lichtbüschel ganz in die Röhre fallt.

Hierdurch erhält man weniger direktes und mehr reflektiertes Licht. Dadurch wird nicht nur die Beleuchtung weniger stark, sondern auch weniger gleichmässig. Dies erklärt sich folgendermassen: Wenn ein paralleler Lichtbüschel auf einen zylinderförmigen Spiegel fällt, entstehen nach Reflexion die sogenannten kaustischen Linien oder Brennlinien. Dies ist eine Folge der sphärischen Aberration. Diese kaustischen Linien sehen wir z.B. auftreten, wenn wir Sonnenlicht auf einen kreisförmigen spiegelnden Metallreifen fallen lassen, der auf einem Blatt Papier liegt. Die Ursache ist ein verschiedenes Verhalten der Randstrahlen, und der zentralen Strahlen des auf den Spiegel fallenden Lichtbüschels (Schema 3).

Die reflektierten Lichtstrahlen schneiden einander nicht in einem Punkt, sondern entlang einer Linie, die dann eine stärkere Beleuchtung in Bezug auf die Umgebung ergibt.

Um ein Missverständnis zu vermeiden, sei noch erwähnt, dass nicht nur ein paralleler Lichtbüschel nach einer derartigen Reflexion kein punktförmiges Einanderschneiden der reflektierten Lichtstrahlen ergibt, sondern auch andere Formen von Lichtmengen. Jedoch habe ich zur Erläuterung,

einen parallelen Lichtbüschel gewählt, weil das Licht in der Untersuchungsröhre nur sehr wenig gestreut ist, also beinahe die Form des parallelen Lichtbüschels hat.



Schema 3.
S. Hohlspiegel (zylinderförmig).
F. Brennpunkt.
o. Zentrale Strahlen.
p. Randstrahlen.

Beim Gebrauche des HASLINGERschen Elektroskops sehen wir die kaustischen Linien schon stark auftreten, was ein dritter Nachteil dieses Elektroskops ist, besonders in Hinblick auf die direkte Photographie. Bei kürzeren Röhren sehen wir die Brennlinien in der unteren Hälfte des beleuchteten Feldes auftreten, weil die meiste Reflexion gegen die Unterwand stattfindet (Schema 1). Bei längeren Röhren treten diese kaustischen Linien im untersten und im obersten Teil des beleuchteten Feldes auf, da nun auch viel Licht gegen die obere Wand der Röhren reflektiert wird. Entsteht nun, wie dies bei Verlängerung des Spiegels *b* der Fall wäre, weniger direktes und mehr reflektiertes Licht infolge des grösseren Winkels, den der Lichtbüschel mit den Röhren bildet, dann werden die kaustischen Linien intensiver auftreten. In einem solchen Falle werden wir also eine sehr stark ungleichmässige Beleuchtung erhalten und schliesslich natürlich auch eine schwächere Beleuchtung.

Nun kann man sich fragen, warum der Spiegel beim optischen System in Bezug auf die Linse nicht höher angebracht ist. Hierdurch wäre es möglich, dass man das ganze optische System tiefer placieren kann, ohne in das Niveau der Röhre zu kommen, wodurch schliesslich eine grössere Menge Licht direkt auf das zu besichtigende Feld fällt. In diesem Falle aber wird mehr Gegenlicht nach mehreren Nebenachsen der Linse *a*, und zwar nach Nebenachsen mit mehr wagerechtem Verlauf auf das Gesicht des Untersuchers gerichtet sein, und wird fraglos Gegenlicht auf den Augapfel fallen.

Um nun obengenannte Uebelstände zu beseitigen, habe ich eine neue Optik konstruiert, allerdings mit Beibehaltung des Prinzips des kreuzenden Lichtes. Bevor ich zur Beschreibung dieses Instrumentes übergehe, mit

dem es möglich ist, durch die endoskopische Röhre photographische Momentaufnahmen zu machen, möge erst der heutige Stand der Endophotographie von Hals- und Brusteingeweiden näher erörtert werden. Von den Luftwegen konnte bisher nur der Larynx photographiert werden und zwar in der Weise, dass das Bild aufgenommen wurde, das ein in den Pharynx eingebrachter Spiegel von dem Larynx zurückwarf (HEGENER, HALL, u.a.). Dieses Verfahren könnte man indirekte Endophotographie nennen. Es ist jedoch nicht möglich, auf diese Weise tiefer gelegene Organteile zu photographieren. Hier sei noch erwähnt, dass Dr. STRUYCKEN in Breda ein Instrument beschrieben hat, mit welchem er auf direkte Weise Photographien vom Kehlkopfeingang anfertigen kann. Derartige Aufnahmen sind jedoch bisher meines Wissens noch nicht veröffentlicht worden.

Mit meinem Instrument als Lichtquelle ist es gelungen, durch die endoskopische Röhre Momentphotographien vom Larynx, von der Verzweigung der Luftröhre in den beiden Hauptbronchien, und von der Bronchialverzweigung des Lungenunterlappens aufzunehmen. Im Gegensatz zu dem vorher Beschriebenen wäre dieses Verfahren als direkte Endophotographie zu bezeichnen.

Durch technische Umstände war es bisher nicht möglich, dieses Verfahren bei Patienten anzuwenden. Die bis jetzt hergestellten Photographien sind an einem vollständigem Larynx-Lungenpräparat eines Erwachsenen, das Herr Professor DEELMAN mir freundlichst zur Verfügung stellte, aufgenommen worden. Die Bedingungen beim Anfertigen der Aufnahmen waren genau dieselben wie in der Klinik. Nur der Umstand, dass der photographische Apparat, über den ich verfüge, bisher nicht an das Endoskop angeschlossen werden konnte, verhinderte, dass auch bei Patienten derartige Aufnahmen gemacht wurden. Da aber die beigefügten Aufnahmen die Möglichkeit des Photographierens durch die endoskopischen Röhren zur Genüge beweisen, meine ich diesen Befund schon jetzt veröffentlichen zu können.

Nachstehend möge die Beschreibung der angewandten Optik folgen.

Bei meinem System (Schema 4) sind die verschiedenen Faktoren hauptsächlich experimentell auf solche Weise bestimmt, dass das System, mit welcher Röhrenlänge man auch arbeitet, immer dieselbe Stellung einnehmen kann, um eine optimale Beleuchtung zu ergeben. Hierbei kommt es nicht nur darauf an, die zur Beleuchtung dienende Lichtmenge möglichst stark, sondern auch die Beleuchtung an sich möglichst gleichmässig zu machen, damit es keine Teile des Feldes gibt, welche zu wenig Beleuchtung bekommen (Schema 4).

Die verschiedenen Faktoren sind: Grösse der Objektdistanz (Entfernung des Glühdrahtes der Beleuchtungslampe von der Optik); Gesamtstärke des Linsensystems, also Grösse der Bilddistanz; Durchschnitt der Linsen, wodurch die Lichtmenge, die in die Röhre tritt und auch der mehr oder weniger schräge Verlauf des Lichtbüschels, also die Reflexion gegen die

Röhrenwand und letzten Endes die Gleichmässigkeit der Beleuchtung bestimmt wird; Prismaform des ganzen Systems. Die Entstehung der

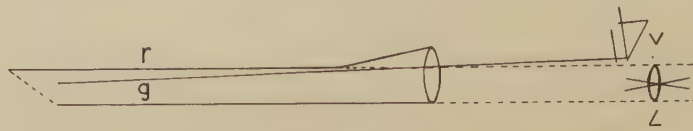
Schema 4.
1, 2, 3. Plan-konvexe Linsen.
4. Flachspiegel.
h. Glühdraht.
g. Hauptachse des Lichtbüschels.
t. Bild des Glühdrahts.

Prismaform hat die folgende Ursache. Im Anfang wurden ganze Linsen gebraucht. Hierbei stellte sich heraus, dass durch die Grösse der Linsen die Mitte der Optik zu weit von der Achsenverlängerung der Untersuchungsröhre entfernt blieb, sodass die Hauptachse des Lichtbüschels einen zu grossen Winkel mit der Achse der Röhre bildete.

Das einfallende Licht musste zu oft reflektiert werden, um seinem Beleuchtungszweck zu entsprechen. Um den Winkel zwischen der Hauptachse des Lichtbüschels und der Röhrenachse zu verkleinern, wurden die Linsen an der unteren Seite so abgeschliffen, dass die Beleuchtung am stärksten und gleichmässigsten war. Eine Grenze wurde diesem Verfahren durch die Verminderung der Lichtmenge des Büschels gesetzt: Lichtmenge und Richtung des Büschels verändern nämlich unter diesen Verhältnissen die Beleuchtungsstärke im entgegengesetzten Sinne und bestimmen auf diese Weise ein maximales Abschleifen des Linsensystems. Der Linsendurchschnitt wurde dadurch kleiner, und dies war die Ursache, dass das Glühlämpchen weiter nach unten reichte als der Unterrand der Linse, und also in das Niveau der Röhre. So ist es begreiflich, dass die Prismaform des ganzen Systems entstand. Herrn VAN ALBADA in Bloemendaal bin ich sehr zu Dank verplichtet für die Angabe der Stärke für jede einzelne Linse, in Zusammenhang mit der grössten Lichtmenge. Auch war Herr VAN ALBADA so freundlich, für mich die Linsen zu schleifen. Schema 5 gibt nun das Ganze wieder.

An dem Instrument ist der Handgriff so eingerichtet, dass bei der Optik auch Röhren gebraucht werden können, wie diese bei dem Elektroskop nach BRÜNINGS üblich sind. Da diese Röhren keine proximale Erweiterung haben, ist es erforderlich, dass der Unterrand der Optik etwas innerhalb des Niveaus der Röhre reicht. Zu diesem zweck ist am Instrument eine

Schraube konstruiert, welche dies ermöglicht. Diese Schraube ist so angebracht, dass man mit der tieferen Stellung der Optik auch einen mehr



Schema 5.

r. Untersuchungsröhre.
v. Prisma (3 plan-konvexe Linsen und 1 Flachspiegel).
g. Hauptachse des Lichtbüschels.
L. Photographisches Objectiv.

wagerechten Verlauf der Hauptachse des Lichtbüschels erhält. Dieses Höher- und Tieferstellen ist von dem Untersucher selbst ohne Hilfe leicht auszuführen. Meiner Meinung nach ist der Umstand, dass die Optik bei nicht zu engen Röhren sehr wenig innerhalb des Röhrenniveaus reicht, kein Hindernis für das gute Sehen des relativ viel weiter gelegenen Organteiles. Dies kann auch in Bezug auf das Einführen von Instrumenten gesagt werden. Bei engeren Röhren, wie sie bei der Kinderbronchoskopie üblich sind, ist dies eine Beschwerlichkeit, und können dann besser die Röhren nach HASLINGER angewandt werden. Obgleich bei diesem Beleuchtungsapparat auch kaustische Linien entstehen, sind diese jetzt aber in deutlich geringerem Masse anwesend, sodass man eine viel gleichmässigere Beleuchtung erhält.

In Schema 5, das den Apparat in Hauptsache wiedergibt, repräsentiert die Linse *L* das photographische Objektiv.

In dieser Weise konnten photographiert werden:

1. der Kehlkopfeingang;
2. die Verzweigung der Luftröhre in beiden Hauptbronchien;
3. Unterlappenbronchienverzweigung.

Die Längen der Röhren waren bezw. 14, 26 und 33 cm.

Schliesslich folgt eine kurze Besprechung der Photographien, wobei darauf hingewiesen wird, dass es sich hier um Photographien eines Kadaverpräparates handelt und nicht eines lebenden Menschen. Hierauf ist eine Verschrumpfung zurückzuführen, die man auf den verschiedenen Aufnahmen sieht.

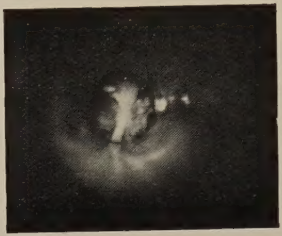
1. Kehlkopfeingang. Wahre Grösse. Aufnahmezeit $^1/_{25}$ Sekunde. Offene Blende: 6,3.

Deutlich sind die wahren Stimmbänder, sowie die durch die Arythaenoidei verursachten Erhöhungen zu sehen.

2. Verzweigung der Luftröhre in den beiden Hauptbronchien. Aufnahmezeit $^1/_{25}$ Sekunde. Offene Blende: 6,3. Vergrösserung der ursprünglichen Aufname bis zur normalen Grösse.

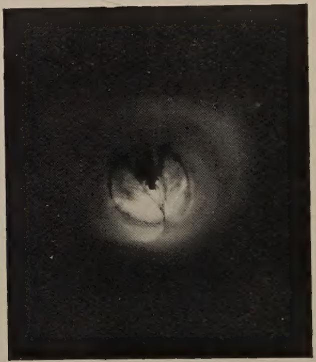
In beiden Bronchien des Präparates befand sich viel eingetrockneter Schleim. Sehr gut ist jedoch die Carina zu sehen.

1.



2.

3.



4.

3. Unterlappenbronchienverzweigung. Aufnahmezeit $^1/_{10}$ Sekunde. Offene Blende: 6,3. Vergrösserung der ursprünglichen Aufnahme bis zur normalen Grösse.

Zum Schluss noch eine hinzugefügte Aufnahme des Kehlkopfes, die mit dem Elektroskop HASLINGERs als Lichtquelle hergestellt wurde.

4. Kehlkopfeingang. Aufnahmezeit $^1/_{25}$ Sekunde. Offene Blende: 6,3.

Wenn man Aufnahme 1 mit Aufnahme 4 vergleicht, so ist zu bemerken, dass bei dieser letzten Aufnahme die Beleuchtung viel ungleichmässiger ist. So sieht man, dass die wahren Stimmbänder, die auf der Aufnahme 1 als weisse Streifen zu erkennen sind, sich auf Aufnahme 4 kaum unterscheiden lassen.

Weiter fällt auf, dass Aufnahme 4 kleiner ist als Aufnahme 1. Dies beruht nicht auf dem Gebrauch anderer Röhren, sondern auf der Tatsache, dass der photographische Apparat, um das Gegenlicht möglichst auszuschalten, weiter von der Röhre entfernt werden musste, wodurch die Objektdistanz grösser, somit die Bilddistanz kleiner und also das Bild kleiner wurde, als dies bei Aufnahme 1 der Fall war. Was bei Aufnahme 4 von dem Gegenlicht noch übrig geblieben war, verursachte eine sehr starke Abnahme der Tiefenschärfe.